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On the Representations of some Infinite Dimensional Groups and Algebras Related to Quantum Physics.

Johnny T. Ottesen

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On the Representations of some Infinite Dimensional Groups and Algebras Related to Quantum Physics

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Abstact.

This is mostly a new version of my Ph.D. thesis including changes and corrections. The paper focus on the spin representation of the restricted orthogonal group and the metaplectic group together with their applications to the loop circle and the group of orientation preserving diffeomorphism on the unit circle. In an attempt to make this paper self-contained we also treate the fermion algebra and the boson algebra, as we include a chapter on the loop algebras and the Virasoro algebra together with a discussion of the various representations thereof. The groups considered are realized as operators acting on an infinite dimensional separable Hilbert space and the various representations are mostly in the different Fock Hilbert spaces or subspaces thereof.

We point out that some of the considered problems originally came from theoretical physics and that this, beyond the nice mathematics it represents, has been some of our motivation for studying these subjects. Therefore we also relate the mathematical theory developed to its natural applications in theoretical physics.

On Representations of some Infinite Dimensional Groups and Algebras Related to Quantum Physics.

Johnny T. Ottesen

May 5, 1992

... thought Alice, and she went on.
"Would you tell me, please, which way I ought to go from here ?"
"That depends a good deal on where you want to get to", said the Cat.
"I don't much care where -" said Alice.
"Then it dosen't matter which way you go", said the Cat.
"- so long as I get somewhere", Alice added as an explanation.
"Oh, you're sure to do that", said the Cat, "if you only walk long enough".

Alice's Adventures.
Lewis Carroll.

Introduction.

Recently there has been a renewed interest in representations in connection with the study of Kac-Moody algebras, loop algebras, the Virasoro algebra and the associated groups ([P-S], [K-R],[Lu], [G-W], [Ar] and many others). These groups appear naturally in quantum theory with infinitely many degrees of freedom, i.e. in canonical quantum field theory, string theory, statistical quantum physics and soliton theory. The renewed interest comes partly from the many fruitful applications in physics and partly, or rather basicaly, from the mathematical success in dealing with infinite dimensional Lie algebras.

Some of the problems in dealing with infinite dimensional Lie algebras are that many of the essential methods known from finite dimensional Lie algebras can not be suitably generalized. In particular such operations as exponentiating elements of the Lie algebras are not always possible, even locally ([G-W] and [P-S]).

The loop group is the group of smooth mappings from the unite circle S^1 to a Lie group G, and it is denoted LG (sometimes $Map(S^1, G)$ or S^1G). If the Lie algebra of the Lie group G is $\mathfrak{g}$, then the Lie algebra of LG is $S^1\mathfrak{g}$ (sometimes $l\mathfrak{g}$ or $Map(S^1,\mathfrak{g})$), see for example [P-S]. The Lie algebra $S^1\mathfrak{g}$ is called the loop algebra of $\mathfrak{g}$. Hereby the loop algebra gives imformation on the loop group. The central extension of the loop algebra $S^1\mathfrak{g}$ is denoted the affine Kac-Moody algebra associated with the Lie algebra $\mathfrak{g}$. Sometimes one need a further extension of an affine Kac-Moody algebra, this is also called an affine Kac-Moody algebra or just an affine algebra ([K-R]), we prefer the last nomenclature.

From a mathematical point of view, the Virasoro algebra is a relatively nice infinite dimensional Lie algebra, and its representations are well understood. The Virasoro algebra is the central extension of the complexification of the smooth real vector fields on the unit circle S^1, hence it is a central extension of the complexification of the Lie algebra of the diffeomorphism group. The exponential mapping from the Lie algebra of real smooth vector fields on the unit circle to the diffeomorphism group is neither locally one-to-one nor onto ([P-S]). It turns out that the diffeomorphism group on the unit circle $Diff(S^1)$ acts as a group of automorphisms on any loop group and that the orientation preserving subgroup $Diff^+(S^1)$ acts projectively on all the known representations of loop groups ([P-S]). Hence it is natural to study the Virasoro algebra in connection with the loop algebras.

The loop algebras and the Virasoro algebra combine in the Sugawara construction, which defines the generators of a representation of the Virasoro algebra as quadratic expressions in the basis of any representation of the affine algebra corresponding to the mentioned loop algebra. This turns out to be very important in conformally invariant statistical physics ([B-P-Z 1] and [B-P-Z 2]).

In a standard approach to quantum field theory the states of the physical system are vectors in a Fock Hilbert space. The precise structure of the Fock Hilbert space depends on the type of statistics obeyed by the particles it describes. In case of bosons one uses the symmetric tensor algebra, $\mathcal{F}_\vee(\mathcal{H})$, modelled over a "one-particle Hilbert space" $\mathcal{H}$, to describe the physical many particle states, hence it is sometimes called the boson Fock Hilbert space. Dealing with fermions, the physical many particle states is described by the Hilbert space completion, $\mathcal{F}_\wedge(\mathcal{H})$, of the exterior algebra over a one-particle Hilbert space $\mathcal{H}$, it is sometimes denoted the fermion Hilbert space. The basic one-particle Hilbert space will, in our considerations, always be separable and infinite dimensional.

The symmetry of the Fock Hilbert spaces (or in a physical language of the particles) are reflected in the commutation relations. In the fermion case, the abstract creation and annihilation operators, which generates a C^*-algebra, parametrized by vectors in a Hilbert space, fulfils the canonical anti-commutation relations, *CAR*. This C^*-algebra is called the *CAR*-algebra or sometimes the fermion algebra. It has a very useful realization on the anti-symmetric Fock Hilbert space, called the Fock representation of the *CAR*-algebra. In the case of bosons, the abstract creation and annihilation operators generates (only) a *-algebra parametrized by vectors in a Hilbert space and fulfils the canonical commutation relations, *CCR*. This *-algebra is sometimes denoted the boson algebra. However, the boson algebra gives rise to the Weyl operators, which are unitaries constructed from the creation and annihilation operators, and they generate a C^*-algebra, called the *CCR*-algebra. Similar to the anti-symmetric case, there are very useful realizations of the boson algebra and the *CCR*-algebra in the symmetric Fock Hilbert space, called the Fock representation of the boson algebra and the *CCR*-algebra, respectively. It turns out that both the fermion and the boson algebras are unique up to isomorphism, whereby we can (and will) benefit from the Fock representations of these in the study of certain automorphism groups.

The automorphism groups mentioned above are, under certain circumstance, unitarily implemented in the Fock representation. These circumstances are closely related to the restricted orthogonal group and the restricted symplectic group, in the case of the fermion algebra and the boson algebra, respectively. The unitary implementers can be constructed explicitly as projective representations of the restricted orthogonal group and the restricted symplectic group, respectively. It turns out that the loop group, LS^1, can be realized as a subgroup of the restricted orthogonal group and that the orientation preserving diffeomorphism group can be realized both as a subgroup of the restricted symplectic group and of the restricted orthogonal group. This will be used to construct particular representations of these groups on a Lie algebra level.

This thesis is an attempt to give a self-contained review of the spin representation of the infinite dimensional restricted orthogonal group and, quite similar, of the metaplectic representation of the infinite dimensional restricted symplectic group together with their connections to the fermion algebra and the boson algebra, respectively. Moreover we clarify the connection of these topics with the theory of loop algebras, Kac-Moody algebras and the Virasoro algebra, which we also discuss in a self-contained menner. Especially, we apply the representations to the particular loop group LS^1 and the diffeomorphism group $\mathit{Diff}^+(S^1)$. Furthermore, we construct various Fock Hilbert space representations of the Virasoro algebra.

Our major tool in dealing with these topics is functional analysis, working mostly on a Lie algebra level. We emphasize that we only consider separable infinite dimensional Hilbert spaces, hence the groups and algebras appearing will also be of infinite dimension.

We will now outline the contents of this thesis systematically. Each section begins with some coments. Some are historical remarks others are an attempt to explain the connection between the particular section and the rest of the contents. Moreover, we bring a list of references on which the particular section is founded, how will sometimes be explained there but sometimes the explanation will appear in the content.

Chapter 1 is a survey of mostly known stuff, however, we have rewritten the material in a collected form. The results of section 1.6 do not, to our knowledge appear anywhere else.

In *section* 1.1, we introduce the Fock Hilbert spaces together with some of its important subspaces. The explicit constructions will be used intensively in this and the following chapters.

In *section* 1.2, we describe the, up to *-isomorphism, unique C^*-algebra called the CAR-algebra over a Hilbert space and its Fock representation. Here CAR stands for canonical anti-commutation relations. Sometimes the CAR-algebra is also called the fermion algebra. The CAR-algebra has an equivalent formulation in terms of the Clifford algebra, which we also discuss. Since the CAR-algebra, and therefore the Clifford algebra, too, are "unique", we will analyse them in their Fock representations. Moreover, we state the question: For which orthogonal operators is the automorphism, $\pi(f) \rightarrow \pi(T^{-1}f)$, of the Clifford algebra, unitarily implementable in the Fock representation ?

In *section* 1.3, we use the explicit construction of the anti-symmetric Fock Hilbert space and the Fock representation of the CAR-algebra to discuss the construction of second quantization (on the anti-symmetric Fock Hilbert space). This second quantization is essential for our construction of the spin representation, treated in the next section.

In *section* 1.4, we construct the so called spin representation of the infinite dimensional restricted orthogonal group, on a Lie algebra level, by use of the second quantization in the anti-symmetric Fock Hilbert space and some generalizations of this ideas. This discussion is closely related to the answer of the question stated in section 1.2, since the automorphism, $\pi(f) \rightarrow \pi(T^{-1}f)$, of the Clifford algebra, is unitarily implementable in the Fock representation if and only if the orthogonal transformation T belongs to the restricted orthogonal group $\mathcal{O}_2(\mathcal{H})$. In fact, this answer was given by Shale and Stinespring in 1965. However, our methods are quite different from theirs, in that we construct the representation explicitly on the subgroup consisting of one-parameter groups in a neighbourhood of the identity in $\mathcal{O}_2(\mathcal{H})$. We finally end this section by giving an explicit formula for the vacuum functional. Beyond giving an explicit projective representation of the restricted orthogonal group, this section will be used, in chapter 4 of applications, to make an explicit representation of the orientation preserving diffeomorphism group, $\mathit{Diff}^+(S^1)$, on the unit circle, on a Lie algebra level, by realising $\mathit{Diff}^+(S^1)$ as a subgroup of the (enlarged) restricted orthogonal group on a particular Hilbert space.

In *section* 1.5, we consider the restriction of the spin representation to the infinite dimensional resricted unitary group, viewed as a subgroup of the restricted orthogonal group. There are mainly two reasons for doing this. It is interesting in itself to have a projective representation of the restricted unitary group, especially does the expression for the Lie algebra cocycle become relatively simple. Moreover, we will use the theory derived in this section, in chapter 4 of applications, to construct a representation in the anti-symmetric Fock Hilbert space of the particular loop group LS^1. Furthermore, we will use the restriction of the spin representation to the restricted unitary group to construct a representation of $Diff^+(S^1)$ considered as a subgroup of the (enlarged) restricted unitary group, on a Lie algebra level.

In *section* 1.6, we give the transformation between the particular formalisms we use, the formalism that deals with, what Vershik calls, almost linear operators, and the formalism that for example Araki uses. The advantage of our notation is that we are dealing with the physical Hilbert space. However, the operators considered become orthogonal operators. In contrast to this, the other point of view, which Araki represents, prefers to deal with a Hilbert space which is twice as big as the physical one, but the advantage is that the orthogonal operators, in our formalism, become unitarities which commute with a certain complex conjugation operator, in this formalism.

Chapter 2 is devoted to the corresponding symmetric case of that treated in chapter 1. We follow the pattern outlined in chapter 1. Our contribution is again the rewriting of the material in these elaborated and collected form, together with the details in section 2.3, which do not appear anywhere else, although the results are well-known.

In *section* 2.1, we consider the object similar to the CAR-algebra in its Fock representation, however this object suffers from a lack of norm continuity, hence it is not a C^*-algebra, but only a *-algebra, called the boson algebra. The "exponentialization", by Stones theorem, of the self-adjoint closure of the sum of the creation and annihilation operators in the Fock representation, gives the unitary operators called Weyl operators. These Weyl operators generates the C^*-algebra called the CCR-algebra. The dicussion of these section will be used in the following section.

In *section* 2.2, section 2.1 will be used in the construction of second quantization in the symmetric Fock Hilbert space, by methods similar to those of section 1.3. The construction in this section will be used in an essential way in the construction of the metaplectic representation, discussed in next section.

In *section* 2.3, the infinite dimensional symplectic group is introduced and we construct the metaplectic representation of the restricted metaplectic group, on a Lie algebra level, by use of second quantization and some ideas similar to it. It turns out that we may use a strategy similar to that outlined in section 1.4. In particular we state the question: For which symplectic transformations, S, is the automorphism, $\pi(f) \to \pi(S^{-1}f)$, unitarily implementable in the Fock representation ? Similar to the orthogonal case this question has already been answered. Shale's answer from 1962 says that the automorphism is unitarily implementable if and only if S belongs to the restricted symplectic group $\mathcal{S}p_2(\mathcal{H})$. As in the former case our proof is constructive. In fact, these unitary implementers are given by the metaplectic representation of the restricted symplectic group. Beside giving us a explicit projective representation of the restricted symplectic group, this section will be used in chapter 4 of applications, to discuss $\mathit{Diff}^+(S^1)$ realized as a subgroup of the (enlarged) restricted symplectic group, on a Lie algebra level. We end this section by deriving an explicit formula for the vacuum functional in this case.

Chapter 3, is devoted to a general review of the loop algebras and the Virasoro algebra together with certain representations of the Virasoro algebra, which to our knowledge not have been done before in details.

In *section* 3.1, we introduce the loop algebra in general. One reason for doing this is to obtain knowledge on loop groups, of which we study a particular one in chapter 4 of applications. Central extension of loop algebras are very important examples of affine Kac-Moody algebras. It is a fact that the simplest representations of affine Kac-Moody algebras are given in terms of the spin representation, whereby the connection of this section and chapter 1 is clarified. Another reason for studying loop algebras is that the diffeomorphism group $\mathit{Diff}^+(S^1)$, to be studied in chapter 4 of applications, acts as a group of automorphisms of the central extesion of loop algebras, or equivalently at the group level, that the diffeomorphism group $\mathit{Diff}^+(S^1)$ acts as a group of automorphisms of the central extension of loop groups. Moreover, we end this section by describing how the loop algebras are related to the Virasoro algebra, which is the central extension of the Lie algebra $\mathit{Vect}(S^1)$ of $\mathit{Diff}^+(S^1)$, of smooth vector fields on the unit circle, namely through the Sugawara construction. It simply gives a representation of the Virasoro algebra by expressing its generators as quadratic terms in the basis elements of any representation of an affine Kac-Moody algebra.

In *section* 3.2, we discuss the Virasoro algebra mentioned above. Its relation to the loop algebras and groups is described in the former section. In chapter 4 of applications we construct some representations of the Virasoro algebra by use of the spin representation and the metaplectic representation, respectively. However, before doing this, we will, in the next section, discuss some concrete representations of the Virasoro algebra.

In *section* 3.3, we construct some explicit representations of the Virasoro algebra with central charge $c = \frac{1}{2}$, in the anti-symmetric Fock Hilbert space. It turns out that there are two essential different cases: In the Ramond sector and in the Neveu-Schwarz sector, respectively. These are irreducible unitary lowest weight representations of positive finite energy. The construction is also an illustration on how one in principle can construct a lot of representations of the Virasoro algebra for central charge larger than one (and, as here for $c = \frac{1}{2}$), however, these cases become a bit simpler. It shall be pointed out that the case $c = \frac{1}{2}$ is of special interest since it is closely related to the complicated Ising model.

In *section* 3.4, we will construct a serie of representations of the Virasoro algebra with central charge belonging to the discrete serie $c_m = 1 - \frac{6}{(m+2)(m+3)}$, where $m \in \mathbf{N} \cup \{0\}$, all in $[0, 1[$. This, purely algebraic, construction is known as the Goddard-Kent-Olive construction. As in the former sections we approach the method by more analytical means, i.e. we expand the known method to constructing representations in a Hilbert space, namely the anti-symmetric Fock Hilbert space. In particular do we obtain a representation of the Virasoro algebra with central charge $c = c_m = \frac{1}{2}$, for $m = 1$.

Chapter 4, deals with applications of the former chapters and thereby it connects the topics considered earlier. The general discussions of the loop algebras and the Virasoro algebra will especially serve as a background for this chapter. Many of the aspects appearing in this chapter has been considered shortly by Lundberg in [Lu 2], we elaborate these considerations and add some too, which to our knowledge have not been published previously.

In *section* 4.1, we use the basic knowledge of loop groups derived in section 3.1 in discussing the particular loop group LS^1, also called the loop circle. It turns out that it can be decomposed into a product of the special loop group SLS^1 and the charge group. We will realize SLS^1 as an abelian subgroup of the restricted unitary group and then use the spin representation to make explicit representations of SLS^1, on a Lie algebra level. It follows that the spin representation of the special loop group SLS^1 fulfils the Weyl form of the canonical commutation relations and thereby becomes a representation of the CCR-algebra, in the anti-symmetric Fock Hilbert space (indicating the socalled boson-fermion correspondance). Due to the fact that the elements of the Lie algebra of the charge group fail to be Hilbert-Schmidt when commuted with a certain projection operator, it follows that we cannot use the method developed in section 1.4 or 1.5. So we have to discuss the charge group by other means. We can nevertheless explicitly construct unitary implementers, whence the product of the representation of SLS^1 and the unitary implementers for the charge group, provide us with explicit projective representations of the particular loop group LS^1.

In *section* 4.2, we study the orientation preserving diffeomorphism group, by realizing it as a subgroup of the restricted unitary group, using the spin representation of the restricted unitary group, on a Lie algebra level. Thereby we get a serie of positive energy representations of the Virasoro algebra, with central charge $c = 1$, in terms of the spin representation on the anti-symmetric Fock Hilbert space.

In *section* 4.3, we make considerations analogous to those of section 4.2. However, this time we will consider $\mathit{Diff}^+(S^1)$ as a subgroup of the symplectic group on a particular Hilbert space and thereby we may use the metaplectic representation to obtain a projective positive energy representation, on a Lie algebra level, of the Virasoro algebra with central charge $c = 1$ and lowest weight $h = 0$.

In *section* 4.4, we cosider the boson-fermion correspondance, not in general, but in the case discussed in section 4.1. The reason why we have delayed this discussion until now is that we will use some of the considerations arising naturally in the former sections of this chapter.

Chapter 5, is reserved for summary, conclusions and outlook.

Chapter 6, contains a summary in danish. At the end we have placed a *Bibliography.*

I want to aknowledge the *Department of Mathematics and Physics and their Functions in Education, Research and Applications* at Roskilde University Centre for giving me the opportunity to write this thesis. Especially I wish to thank my supervisor Anders Madsen and Lars-Erik Lundberg, Department of Mathematics, University of Copenhagen, for fruitful discussions and lots of advice.

Johnny T. Ottesen

Contents

Chapter 1

The Spin Representation of the Infinite Dimensional Orthogonal (Metagonal) Group based on the CAR Algebra in its Fock Representation

1.1 The Fock Hilbert spaces.

In this section we will describe the construction and interpretation of the Fock Hilbert spaces together with some of its important subspaces. It will be used intensively in the following.

It was the physicist V. Fock who introduced the Fock Hilbert spaces, as a working ground of quantum field theory, in 1932 ([Fo]), but it was F. Murray and J. von Neumann who first gave a detailed description of finite tensor products of Hilbert spaces, in 1936 ([M-N]), though tensor products of finite dimensional spaces were known long before that. This section is based on [R-S 1], [Fo], [M-N] and [B-R 2].

Let $\mathcal{H}$ denote a separable complex Hilbert space, with inner product $\langle\cdot,\cdot\rangle$, complex linear in the right hand argument. The vectors in $\mathcal{H}$ describe *the physical one-particle states* of a given quantum physical system. Let $\otimes^n\mathcal{H}$ denote the n-fold (Hilbert space) tensor product of n identical copies of $\mathcal{H}$, for $n \in \mathbf{N}$, where $\otimes^1\mathcal{H}$ is identified with $\mathcal{H}$. We define $\otimes^0\mathcal{H} = \mathbf{C}$. Thus the vectors in $\otimes^n\mathcal{H}$, $n \in \mathbf{N}$, describes *the n-particle quantum physical states*, hence $\otimes^n\mathcal{H}$ is called *the n-particle space.* We now define *the Fock Hilbert space* $\mathcal{F}(\mathcal{H})$ as the (Hilbert space) direct sum of the n-fold tensor products $\otimes^n\mathcal{H}$, $n \in \mathbf{N}\cup\{0\}$, i.e.

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}(\otimes^n\mathcal{H})$$

providing a canonical grading of $\mathcal{F}(\mathcal{H})$. Thus a vector $F = \oplus_{n=0}^{\infty}F_n$ of $\mathcal{F}(\mathcal{H})$ can be viewed as a sequence $\{F_n\}_{n=0}^{\infty}$ of vectors $F_n \in \otimes^n\mathcal{H}$ such that $\sum_{n=0}^{\infty}\|F_n\|^2 < \infty$, where the norm is taken in the respective spaces, and

$$\|F_n\| = \|f_1 \otimes \cdots \otimes f_n\| = \|f_1\| \cdot \ldots \cdot \|f_n\|$$

for $F_n \in \otimes^n\mathcal{H}$ of the form $F_n = f_1 \otimes \cdots \otimes f_n$, with $f_1, \ldots, f_n \in \mathcal{H}$. Notice that $\mathcal{F}(\mathcal{H})$ is a Hilbert space with the natural inner product given by $\langle F, G\rangle = \sum_{n=0}^{\infty}\langle F_n, G_n\rangle$, where $F = \{F_n\}_{n=0}^{\infty}$, $G = \{G_n\}_{n=0}^{\infty}$ and $\langle F_n, G_n\rangle = \langle f_1, g_1\rangle \cdot \ldots \cdot \langle f_n, g_n\rangle$ on product vectors $F_n = f_1 \otimes \cdots \otimes f_n$ and $G_n = g_1 \otimes \ldots \otimes g_n$. The linear span of such product vectors in $\otimes^n\mathcal{H}$, $n \in \mathbf{N}\cup\{0\}$ forms a dense set in $\mathcal{F}(\mathcal{H})$, by definition. We write $F = \oplus_{n=0}^{\infty}F_n$ and $F = \{F_n\}_{n=0}^{\infty}$ interchangeable. The n-particle Hilbert space $\otimes^n\mathcal{H}$ can be identified with the closed subspace of $\mathcal{F}(\mathcal{H})$, consisting of vectors $F = \{F_n\}_{n=0}^{\infty}$ with all components except the n'th equal to zero. Frequently we will consider the closed subspace $\mathcal{D}$ of $\mathcal{F}(\mathcal{H})$ consisting of vectors $F = \oplus_{n=0}^{\infty} F_n$ with only finite many non-zero components F_n, i.e. $\mathcal{D}$ is the algebraic direct sum of the $\otimes^n \mathcal{H}$, and $\mathcal{F}(\mathcal{H})$ is the completion of $\mathcal{D}$ with respect to the Hilbert space norm arising from the inner product. Finally we define the closed subspace $\mathcal{D}^{(k)}$ of $\mathcal{F}(\mathcal{H})$ consisting of vectors $F = \oplus_{n=0}^{\infty} F_n$ such that F_n is zero for $n > k$, hence we can identify $\mathcal{D}^{(k)}$ with $\oplus_{n=0}^{k} \otimes^n \mathcal{H}$. Notice that $\mathcal{D}^{(k)}$ is also a closed subspace of $\mathcal{D}$. The distinguished vector $\Omega = \oplus_{n=0}^{\infty} \Omega_n$ with $\Omega_0 = 1$ and $\Omega_n = 0$ for all $n \in \mathbf{N}$, is called the *vacuum vector*, since it describes the "empty space" corresponding to no particles.

Actually it is not $\mathcal{F}(\mathcal{H})$ itself, but rather two of its closed subspaces, described below, which are used most frequently. Define two operators on $\otimes^n \mathcal{H}$, *the symmetrization operator* $P_\vee$ and *the anti-symmetrization operator* $P_\wedge$ given on product vectors $f_1 \otimes \cdots \otimes f_n \in \otimes^n \mathcal{H}$ by

$$P_{\text{⩖}}(f_1 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\sigma \in \mathcal{S}_n} \chi_{\text{⩖}}(\sigma) \cdot f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$$

where $P_{\text{⩖}}$ denote either $P_\vee$ or $P_\wedge$ and the corresponding index on $\chi_{\text{⩖}}(\sigma)$ follows that of $P_{\text{⩖}}$. The sum is taken over all permutations σ in the permutation group $\mathcal{S}_n$ of permutations of n elements and $\chi_\wedge(\sigma)$ is the sign of the permutation σ, whereas $\chi_\vee(\sigma)$ is constant 1, and could be omitted. One easily checks, by direct calculations, that both operators $P_{\text{⩖}}$ in fact are orthogonal projections. We introduce the notation

$$\begin{aligned} f_1 \wedge \cdots \wedge f_n &= (n!)^{\frac{1}{2}} P_\wedge(f_1 \otimes \cdots \otimes f_n) \\ &= (n!)^{-\frac{1}{2}} \sum_{\sigma \in \mathcal{S}_n} \chi_\wedge(\sigma) \cdot f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \end{aligned}$$

and its symmetric analogue

$$\begin{aligned} f_1 \vee \cdots \vee f_n &= (n!)^{\frac{1}{2}} P_\vee(f_1 \otimes \cdots \otimes f_n) \\ &= (n!)^{-\frac{1}{2}} \sum_{\sigma \in \mathcal{S}_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \end{aligned}$$

Moreover we write $\wedge^n \mathcal{H}$ for $P_\wedge(\otimes^n \mathcal{H})$ and $\vee^n \mathcal{H}$ for $P_\vee(\otimes^n \mathcal{H})$, which clearly are closed subspaces of $\otimes^n \mathcal{H}$. These orthogonal projections are extended in an obvious way to the whole Fock Hilbert space, denoted by the same symbols, by putting

$$P_{\text{⩖}}(\oplus_{n=0}^{\infty} F_n) = \oplus_{n=0}^{\infty} (P_{\text{⩖}} F_n) \tag{1.1}$$

for $F_n \in \otimes^n \mathcal{H}$. In fact this is done by first extending $P_{\varkappa}$ by linearity, giving two densely defined operators, both with norm 1, then followed by an extension by continuity to bounded operators on $\otimes^n \mathcal{H}$, and finally followed by the extension given above, in formula (1.1). We will write $\mathcal{F}_{\varkappa}(\mathcal{H})$ for $P_{\varkappa}\mathcal{F}(\mathcal{H})$, respectively. Formula (1.1) gives us a natural grading of $\mathcal{F}_{\varkappa}(\mathcal{H})$. Since any product vector of the form $f_1 \vee \cdots \vee f_n$, with $f_1, \ldots, f_n \in \mathcal{H}$, is totally symmetric, i.e. are invariant under all permutations of the vectors $f_1, \ldots, f_n \in \mathcal{H}$, we call $\mathcal{F}_{\vee}(\mathcal{H})$ *the symmetric Fock Hilbert space* over $\mathcal{H}$ or *the boson Fock Hilbert space*, reflecting the fact that its physical states (vectors) describe bosons (particle obeying Bose-Einstein statistics). Since any product vector of the form $f_1 \wedge \cdots \wedge f_n$, with $f_1, \ldots, f_n \in \mathcal{H}$, is anti-symmetric, i.e. an interchange of any two one-particle vectors in the product vector is equal to minus the original product vector, we call $\mathcal{F}_{\wedge}(\mathcal{H})$ *the anti-symmetric Fock Hilbert space* over $\mathcal{H}$ or *the fermion Fock Hilbert space*, reflecting the fact that its physical states describe fermions (particle obeying Fermi-Dirac statistics).

Alternatively one could define $\mathcal{F}_{\wedge}(\mathcal{H})$ as the Hilbert space completion of the exterior algebra over the complex Hilbert space $\mathcal{H}$ and $\mathcal{F}_{\vee}(\mathcal{H})$ as the Hilbert space completion of the symmetric tensor algebra over the complex Hilbert space $\mathcal{H}$.

The closed subspaces $\mathcal{D}$ and $\mathcal{D}^{(k)}$, discussed earlier, give rise to the corresponding closed subspaces $\mathcal{D}_{\varkappa} = \mathcal{D} \cap \mathcal{F}_{\varkappa}(\mathcal{H})$ and $\mathcal{D}_{\varkappa}^{(k)} = \mathcal{D}^{(k)} \cap \mathcal{F}_{\varkappa}(\mathcal{H})$, in $\mathcal{F}_{\vee}(\mathcal{H})$ respectively $\mathcal{F}_{\wedge}(\mathcal{H})$. With abuse of notation we suppress the index and simply write $\mathcal{D}$ and $\mathcal{D}^{(k)}$, respectively, when it is clear from the context in which space we are working.

Before ending this section we will discuss the orthonormal basis of the repective spaces.

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then it follows that

$$\{e_{k_1} \otimes \cdots \otimes e_{k_n} : k_1, \ldots, k_n \in \mathbf{N}\}$$

form an orthonormal basis for $\otimes^n \mathcal{H}$, $n \in \mathbf{N}$, with respect to the inner product given earlier on product vectors as

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \ldots \otimes g_n \rangle = \prod_{i=1}^{n} \langle f_i, g_i \rangle$$

Hence

$$\{\Omega\} \cup \{e_{k_1} \otimes \cdots \otimes e_{k_n} : k_1, \ldots, k_n \in \mathbf{N}, n \in \mathbf{N}\}$$

is an orthonormal basis for $\mathcal{F}(\mathcal{H})$, where we have identified $e_{k_1} \otimes \cdots \otimes e_{k_n}$ with the vector $E = \oplus_{m=0}^{\infty} E_m \in \mathcal{F}(\mathcal{H})$ such that $E_m = 0$ for all $m \in \mathbf{N} \setminus \{n\}$ and $E_n = e_{k_1} \otimes \cdots \otimes e_{k_n}$.

The restriction of the inner product on $\otimes^n \mathcal{H}$ to $\wedge^n \mathcal{H}$ gives, on product vectors $f_1 \wedge \cdots \wedge f_n$ and $g_1 \wedge \cdots \wedge g_n$, the following

$$\begin{aligned}
&\langle f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_n \rangle \\
&= n! \langle P_\wedge (f_1 \otimes \cdots \otimes f_n), P_\wedge (g_1 \otimes \ldots \otimes g_n) \rangle \\
&= \sum_{\sigma \in \mathcal{S}_n} \chi_\wedge(\sigma) \langle f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}, g_1 \otimes \ldots \otimes g_n \rangle \\
&= \sum_{\sigma \in \mathcal{S}_n} \chi_\wedge(\sigma) \prod_{i=1}^{n} \langle f_{\sigma(i)}, g_i \rangle \\
&= \det(\langle f_j, g_i \rangle)_{i,j=1,\ldots,n}
\end{aligned}$$

where $\det(\langle f_j, g_i \rangle)_{i,j=1,\ldots,n}$ denote the determinant of matrix $(\langle f_j, g_i \rangle)_{i,j=1,\ldots,n}$ and we have used that $P_\wedge$ is an orthogonal projection on $\otimes^n \mathcal{H}$. It follows that

$$\{e_{k_1} \wedge \cdots \wedge e_{k_n} : k_1 < \cdots < k_n, k_1, \ldots, k_n \in \mathbf{N}\}$$

form an orthonormal basis for $\wedge^n \mathcal{H}$. Hence

$$\{\Omega\} \cup \{e_{k_1} \wedge \cdots \wedge e_{k_n} : k_1 < \cdots < k_n, k_1, \ldots, k_n \in \mathbf{N}, n \in \mathbf{N}\}$$

is an orthonormal basis for $\mathcal{F}_\wedge(\mathcal{H})$, where we have used the canonical identification of vectors in $\wedge^n \mathcal{H}$ with vectors in $\mathcal{F}_\wedge(\mathcal{H})$, in analogy with above.

The restriction of the inner product to $\mathcal{F}_\vee(\mathcal{H})$ gives

$$\begin{aligned}
\langle f_1 \vee \cdots \vee f_n, g_1 \vee \cdots \vee g_n \rangle &= n! \langle P_\vee (f_1 \otimes \cdots \otimes f_n), P_\vee (g_1 \otimes \ldots \otimes g_n) \rangle \\
&= \sum_{\sigma \in \mathcal{S}_n} \langle f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}, g_1 \otimes \ldots \otimes g_n \rangle \\
&= \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \langle f_{\sigma(i)}, g_i \rangle
\end{aligned}$$

known as the permanent of matrix $(\langle f_j, g_i \rangle)_{i,j=1,\ldots,n}$, where we have used that $P_\vee$ is an orthogonal projection on $\otimes^n \mathcal{H}$. If $(e_{k_1}, \ldots, e_{k_n}) = (e_{\sigma(l_1)}, \ldots, e_{\sigma(l_n)})$ for exactly m different permutations $\sigma \in \mathcal{S}_n$, where $m \in \mathbf{N} \cup \{0\}$, then it follows that

$$\langle e_{k_1} \vee \cdots \vee e_{k_n}, e_{l_1} \vee \cdots \vee e_{l_n} \rangle = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \langle e_{k_i}, e_{\sigma(l_i)} \rangle = m$$

If we collect the factors and write $e_{k_i}^{r_i}$ for $\overbrace{e_{k_i} \vee \cdots \vee e_{k_i}}^{r_i\text{-times}}$, then

$$\langle e_{k_1}^{r_1} \vee \cdots \vee e_{k_n}^{r_n}, e_{k_1}^{r_1} \vee \cdots \vee e_{k_n}^{r_n} \rangle = \prod_{i=1}^{n} (r_i!)$$

and

$$\langle e_{k_1}^{r_1} \vee \cdots \vee e_{k_p}^{r_p}, e_{l_1}^{s_1} \vee \cdots \vee e_{l_m}^{s_m} \rangle = 0$$

if $e_{k_1}^{r_1} \vee \cdots \vee e_{k_p}^{r_p} \neq e_{l_1}^{s_1} \vee \cdots \vee e_{l_m}^{s_m}$, where $n = r_1 + \cdots + r_p = s_1 + \cdots + s_m$ and the inner product is taken in $\vee^n \mathcal{H}$. Then

$$\{\prod_{i=1}^{n} (r_i!)^{-\frac{1}{2}} \cdot e_{k_1}^{r_1} \vee \cdots \vee e_{k_n}^{r_n} : k_1 < \cdots < k_n, k_1, \ldots, k_n, r_1, \ldots, r_n \in \mathbf{N}\}$$

form an orthonormal basis for $\vee^n \mathcal{H}$. Hence

$$\{\Omega\} \cup \{K_{(r)_n} \cdot e_{k_1}^{r_1} \vee \cdots \vee e_{k_n}^{r_n} : k_1 < \cdots < k_n, (k)_n, (r)_n \in \mathbf{N}^n, n \in \mathbf{N}\}$$

is an orthonormal basis for $\mathcal{F}_\vee(\mathcal{H})$, where we have used multiindex notation, $(k)_n = (k_1, \ldots, k_n)$ and $(r)_n = (r_1, \ldots, r_n)$ and the canonical identification of vectors in $\vee^n \mathcal{H}$ with the corresponding vectors in $\mathcal{F}_\vee(\mathcal{H})$, in analogy with earlier. The normalization constant is given by

$$K_{(r)_n} = \|e_{k_1}^{r_1} \vee \cdots \vee e_{k_n}^{r_n}\|^{-1} = \prod_{i=1}^{n} (r_i!)^{-\frac{1}{2}}$$

We will sometimes use the notation of multiindex, as mentioned above, writing $(k)_n$ for $(k_1, \ldots, k_n)$ for $n \in \mathbf{N}$, where $k_i \in \mathbf{N}$ or sometimes $k_i \in \mathbf{N} \cup \{0\}$ for $i = 1, \ldots, n$.

With the intensive knowlege on the structure of the Fock Hilbert spaces we are able to discuss the, socalled, second quatization and related topics in great detail, but first we turn to the discussion of the *CAR-algebra* and its Fock representation, where *CAR*-algebra stands for canonical anti-commutation relations.

1.2 The CAR-algebra and its Fock representation.

In this section we will describe the CAR-algebra, which is a C^*-algebra, unique up to $*$-isomorphism. The CAR-algebra were introduced by Jordan and Wigner ([J-W]) in 1928 for the purpose of quantizing the electron field in physics. There is a very useful concrete realization of the CAR-algebra on the anti-symmetric Fock Hilbert space, called the Fock representation, which is the one physisists use mostly. It was also this representation of the CAR-algebra J.M. Cook, a student of I. E. Segal, used in his fundamental, and remarkable detailed, paper ([Co]) in 1953. However the basic ideas goes back to V. Fock ([Fo]), in 1932, and Jordan and Wigner ([J-W]), in 1928.

This section gives us a nice mathematical frame for handling many problems in quantum physics with infinitely many degrees of freedom. The section is based on [Co], [Ar], [B-R 2] and is, in a certain sense, given in a reverse order of that in [B-R 2].

Let $\mathcal{H}$ be a Hilbert space over $\mathbf{C}$ and $\mathcal{A}$ an abstract C^*-algebra with unit. We call $\mathcal{A}$ a *CAR-algebra over* $\mathcal{H}$ if there is an anti-linear mapping $a : \mathcal{H} \longrightarrow \mathcal{A}$ such that $\{a(f) : f \in \mathcal{H}\}$ together with the unit I, generates the C^*-algebra $\mathcal{A}$ and fulfil the *canonical anti-commutation relations*

$$\begin{aligned} [a(f), a(g)]_+ &= 0 \\ [a(f), a(g)^*]_+ &= \langle f, g\rangle \cdot I \end{aligned} \tag{1.2}$$

for all $f, g \in \mathcal{H}$. Here $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathcal{H}$ and $[\cdot,\cdot]_+$ the usual anti-commutator $[A, B]_+ = AB + BA$. We will shortly refer to the canonical anti-commutation relations, given in (1.2), as the CAR.

By use of the CAR, it follows that $f \longrightarrow a(f)$ is an isometry, since

$$(a(f)^*a(f))^2 = a(f)^*\left(\|f\|^2 \cdot I - a(f)^*a(f)\right)a(f) = \|f\|^2 \cdot a(f)^*a(f)$$

so

$$\|a(f)\|^4 = \|(a(f)^*a(f))^2\| = \|f\|^2 \cdot \|a(f)^*a(f)\| = \|f\|^2 \cdot \|a(f)\|^2$$

hence $\|a(f)\| = \|f\|$, where we have used the same norm-symbol to denote to different norms, one act on the C^*-algebra and the other on the Hilbert space.

Moreover it follows that a CAR-algebra is unique up to $*$-isomorphism. Let $\tilde{\mathcal{A}}$ be another CAR-algebra over the same Hilbert space $\mathcal{H}$ and let $\tilde{a} : \mathcal{H} \to \tilde{\mathcal{A}}$ denote the corresponding anti-linear mapping. Define a $*$-morhpism (i.e. a linear, multiplicative mapping conserving the involution)

$$\alpha : \{a(f), a(f)^* : f \in \mathcal{H}\} \to \{\tilde{a}(f), \tilde{a}(f)^* : f \in \mathcal{H}\}$$

by

$$\alpha(a(f)) = \tilde{a}(f)$$

for all $f \in \mathcal{H}$. Clearly α is an isometry on its domain, hence it can be extended, in a well defined manner, to the C^*-algebra $\mathcal{A}$ generated by $\{a(f) : f \in \mathcal{H}\}$. Moreover its range is the C^*-subalgebra of $\tilde{\mathcal{A}}$, generated by $\{\tilde{a}(f), \tilde{a}(f)^* : f \in \mathcal{H}\}$, i.e. the range is all of $\tilde{\mathcal{A}}$. Then the $*$-morphism α is one-to-one and onto, hence a $*$-isomorphism. So we may speak of *the CAR-algebra*, since it is unique up to $*$-isomorphism.

Let U be any bounded complex linear operator on $\mathcal{H}$ and V any bounded complex anti-linear operator satisfying

$$V^*U + U^*V = 0 = UV^* + VU^*$$

and

$$U^*U + V^*V = I = UU^* + VV^* \tag{1.3}$$

If this is the case we say that U and V are *Bogoliubov transformations*. Notice that the adjoint of an anti-linear operator V is defined in a different way than the adjoint of a linear operator, we have $\langle f, Vg\rangle = \langle g, V^*f\rangle = \overline{\langle V^*f, g\rangle}$ for all $f, g \in \mathcal{H}$, for a more detailed discussion we refer to section 1.4. Put $\tilde{a}(f) = a(Uf) + a(Vf)^*$. Then $\tilde{a}(f)$ is evidently bounded for all $f \in \mathcal{H}$. It follows that the mapping $\tilde{a} : f \to \tilde{a}(f)$ is anti-linear, which implies that

$$\begin{aligned}
[\tilde{a}(f), \tilde{a}(g)]_+ &= \langle Uf, Vg\rangle \cdot I + \overline{\langle Vf, Ug\rangle} \cdot I \\
&= \langle g, (V^*U + U^*V)f\rangle \cdot I \\
&= 0
\end{aligned}$$

and that

$$\begin{aligned}
[\tilde{a}(f), \tilde{a}(g)^*]_+ &= \langle Uf, Ug\rangle \cdot I + \langle Vf, Vg\rangle \cdot I \\
&= \langle (U^*U + V^*V)f, g\rangle \cdot I \\
&= \langle f, g\rangle \cdot I
\end{aligned}$$

for all $f, g \in \mathcal{H}$, where we have used the definition of the adjoint of an anti-linear operator V. So $\{\tilde{a}(f) : f \in \mathcal{H}\} \cup \{I\}$ generate a CAR-algebra over $\mathcal{H}$. Hence, because of the uniqueness, up to $*$-isomorphism, of the CAR-algebra, there exists a unique $*$-automorphism α of the CAR-algebra, such that $\alpha(a(f)) = \tilde{a}(f)$, for all $f \in \mathcal{H}$, and in this case is $\alpha^{-1}(\tilde{a}(f)) = a(U^*f) + a(V^*f)^*$, which follows directly by a calculation.

It is easily proved that the CAR-algebra is separable if and only if $\mathcal{H}$ is separable and that the CAR-algebra is simple (for proofs, see [B-R 2, p. 16]). In our case we will only consider infinite dimensional (complex or real) separable Hilbert spaces.

We now turn to *the Fock representation of the CAR-algebra.* For each $f \in \mathcal{H}$ define the linear operator $a_0{}^*(f)$ on the anti-symmetric Fock Hilbert space $\mathcal{F}_\wedge(\mathcal{H})$ by

$$a_0{}^*(f)\Omega = f$$

and on product vectors by

$$a_0{}^*(f)(f_1 \wedge \cdots \wedge f_n) = f \wedge f_1 \wedge \cdots \wedge f_n$$

Notice that $a_0{}^*(f) : \wedge^n\mathcal{H} \to \wedge^{n+1}\mathcal{H}$. Extension by linearity gives a densely defined linear operator on $\mathcal{F}_\wedge(\mathcal{H})$. For $f \neq 0$ choose an orthonormal basis $\{e_i\}_{i\in\mathbf{N}}$ for $\mathcal{H}$ such that $e_1 = \frac{f}{\|f\|}$. Then one observe that $a_0{}^*(e_1) = \|f\|^{-1}a_0{}^*(f)$ defines a partial isometry with support

$$\mathcal{K} = \overline{span}\{\Omega, e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 < i_1 < \cdots < i_n, i_1, \ldots, i_n, n \in \mathbf{N}\}$$

and range $\mathcal{K}^\perp \cap \mathcal{F}_\wedge(\mathcal{H})$, since $a_0{}^*(e_1)(e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}) = 0$ for all $i_2 < \cdots < i_n$ and $\|a_0{}^*(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_n})\| = \|e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_n}\| = 1$ for all $1 < i_1 < \cdots < i_n$. Hence $\|a_0{}^*(f)\| = \|f\|$ for all $f \in \mathcal{H}$. So $a_0{}^*(f)$ is a bounded operator on $\mathcal{F}_\wedge(\mathcal{H})$, then it has a unique bounded ajoint $a_0{}^*(f)^*$,which we denote $a_0(f)$, it is given by

$$a_0(f)\Omega = 0$$

and

$$\begin{aligned} & a_0(f)(f_1 \wedge \cdots \wedge f_n) \\ = \; & \sum_{i=1}^{n}(-1)^{i+1}\langle f, f_i\rangle \cdot f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_n \end{aligned}$$

since

$$\begin{aligned}
&\langle a_0{}^*(g_1)(g_2 \wedge \cdots \wedge g_n), f_1 \wedge \cdots \wedge f_n\rangle \\
&= \langle g_1 \wedge g_2 \wedge \cdots \wedge g_n, f_1 \wedge \cdots \wedge f_n\rangle \\
&= \det\{\langle g_i, f_j\rangle\}_{i,j=1,\ldots,n}
\end{aligned}$$

and expansion after first row gives

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^{i+1}\langle g_1, f_i\rangle \cdot \langle g_2 \wedge \cdots \wedge g_n, f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_n\rangle \\
&= \langle g_2 \wedge \cdots \wedge g_n, \sum_{i=1}^{n}(-1)^{i+1}\langle g_1, f_i\rangle \cdot f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_n\rangle
\end{aligned}$$

which gives the above formula for $a_0(f) = a_0{}^*(f)^*$ on product vectors, and evidently is $0 = \langle a_0{}^*(f)F_n, \Omega\rangle = \langle F_n, a_0(f)\Omega\rangle$, for any $F_n \in \wedge^n\mathcal{H}$ and for all $n \in \mathbf{N} \cup \{0\}$, so $a_0(f)\Omega = 0$. Observe that $\|a_0(f)\| = \|a_0{}^*(f)\| = \|f\|$. Moreover, notice that $a_0{}^*(f) : \wedge^n\mathcal{H} \to \wedge^{n+1}\mathcal{H}$ and that $a_0(f) : \wedge^{n+1}\mathcal{H} \to \wedge^n\mathcal{H}$, hence their names, *creation* and *annihilation operators*, respectively, since they correspond to creation and annihilation of particles in quantum physics. Observe now that

$$a_0(f)a_0{}^*(f) + a_0{}^*(f)a_0(f) = \|f\|^2 \cdot I$$

first on Ω and on product vectors, hence by linearity and continuity on all of $\mathcal{F}_\wedge(\mathcal{H})$. From this it follows, again, that $a_0{}^*(f)$ and $a_0(f)$ are bounded by $\|f\|$, since both terms on the left hand side are positive. Moreover

$$a_0{}^*(f)a_0{}^*(f) + a_0{}^*(f)a_0{}^*(f) = 0$$

for all $f \in \mathcal{H}$, due to the anti-symmetry of product vectors in $\mathcal{F}_\wedge(\mathcal{H})$. The last equation reflects the famous Pauli principle from physics. Taking the adjoint, we get

$$a_0(f)a_0(f) + a_0(f)a_0(f) = 0$$

for all $f \in \mathcal{H}$. Using the polarization identity we get the CAR

$$[a_0(f), a_0{}^*(g)]_+ = \langle f, g\rangle \cdot I$$

and

$$[a_0(f), a_0(g)]_+ = 0 = [a_0{}^*(f), a_0{}^*(g)]_+$$

Hence the concrete C^*-algebra generated by $\{a_0(f) : f \in \mathcal{H}\} \cup \{I\}$ is a representation of the CAR-algebra, called *the Fock representation of the CAR-algebra* and sometimes physisists denote it *the fermionic field algebra over* $\mathcal{H}$, for obvious reasons. Notice that the vacuum vector Ω is a cyclic vector for the Fock representation of the CAR-algebra.

Now, let P be an arbitrary orthogonal projection in $\mathcal{H}$. Choose an orthonormal basis $\{e_i\}_{i\in\mathbf{N}}$ for $\mathcal{H}$ consisting of eigenvectors for P. Then any $f \in \mathcal{H}$ has an expansion as $f = \sum_{i\in\mathbf{N}} \langle e_i, f\rangle e_i$. We may define an operator Γ on $\mathcal{H}$ given by $\Gamma f = \sum_{i\in\mathbf{N}} \overline{\langle e_i, f\rangle} e_i$, where the bar denote ordinary complex conjugation. Hence Γ is a well defined involution on $\mathcal{H}$ and it commute evidently with P. Moreover, Γ is anti-unitary, $\langle \Gamma f, \Gamma g\rangle = \langle g, f\rangle$, for all $f, g \in \mathcal{H}$. That is, for any orthogonal projection P on $\mathcal{H}$ there exists, at least, one anti-unitary involution Γ on $\mathcal{H}$ such that $[P, \Gamma] = 0$. Let Γ be any such anti-unitary involution on $\mathcal{H}$. Putting $U = I - P$ and $V = \Gamma P$ we see that U and V are Bogoliubov transformations

$$U^*U + V^*V = I - P + P = I = UU^* + VV^*$$

and

$$V^*U + U^*V = \Gamma P(I - P) + (I - P)\Gamma P = 0 = UV^* + VU^*$$

and both U and V are selfadjoint. Then it follows that the C^*-algebra generated by

$$a_P(f) = a_0((I - P)f) + a_0{}^*(\Gamma P f)$$

defines a representation of the CAR-algebra in the Fock space, called the quasi-free representation, in terms of the Fock representation. Observe that this represents the physical idea of "filling up the Dirac sea", since $a_P(f_+)\Omega = 0$, for $f_+ \in (I - P)\mathcal{H}$ (representing that the physical states with positive energy are all unoccupied) and $a_P{}^*(f_-)\Omega = 0$, for $f_- \in P\mathcal{H}$ (representing that the physical states with negative energy are all occupied). So Ω corresponds to the filled Dirac sea, in the case of the quasi-free representation. Note that the definition is, consistent for $P = 0$, so the Fock representation is just a special case of this construction. Moreover Ω is a cyclic vector for this representation of the CAR-algebra. Whence we have the following theorem.
Theorem 1 *The representations of the CAR-algebra given by* $a(f) \to a_P(f)$ *are all irreducible, including the Fock representation.*

Proof. This theorem and it's proof can be found in [B-R 2, proposition 5.2.2] in the case of $P = 0$, which we will repeat and use below. Since $a_0(f) = a_P((I-P)f) + a_P^*(-\Gamma Pf)$ and $a_0^*(f) = a_P^*((I-P)f) + a_P(-\Gamma Pf)$, any operator T commuting with $a_P(f)$ and $a_P^*(f)$, $f \in \mathcal{H}$, commutes with $a_0(f)$ and $a_0^*(f)$, and conversely. Therefore we only need to consider the statement for $P = 0$, since it imply the case when $P \neq 0$. Let T commute with $a_0(f)$ and $a_0^*(f)$, i.e. $[T, a_0(f)] = [T, a_0^*(f)] = 0$, Observe that

$$\begin{aligned} T_{n,m} &= \langle a_0^*(f_1)\cdot\ldots\cdot a_0^*(f_n)\Omega, Ta_0^*(g_1)\cdot\ldots\cdot a_0^*(g_m)\Omega\rangle \\ &= \langle T^*\Omega, a_0(f_n)\cdot\ldots\cdot a_0(f_1)a_0^*(g_1)\cdot\ldots\cdot a_0^*(g_m)\Omega\rangle \end{aligned}$$

is zero for $n > m$ and $n, m \in \mathbf{N}$, since we annihilate more particles than we create from Ω. Analogously, we get

$$\begin{aligned} T_{n,m} &= \langle a_0(g_m)\cdot\ldots\cdot a_0(g_1)a_0^*(f_1)\cdot\ldots\cdot a_0^*(f_n)\Omega, T\Omega\rangle \\ &= 0 \end{aligned}$$

for $n < m$. Moreover, for $n = m$, we have $a_0(f_n)\cdot\ldots\cdot a_0(f_1)a_0^*(g_1)\cdot\ldots\cdot a_0^*(g_m)\Omega = b\cdot\Omega$, where

$$\begin{aligned} b &= \langle \Omega, a_0(f_n)\cdot\ldots\cdot a_0(f_1)a_0^*(g_1)\cdot\ldots\cdot a_0^*(g_m)\Omega\rangle \\ &= \langle a^*(f_1)\cdot\ldots\cdot a^*(f_n)\Omega, a^*(g_1)\cdot\ldots\cdot a^*(g_m)\Omega\rangle \end{aligned}$$

for any $n = m \in \mathbf{N}$. So with $c = \langle T^*\Omega, \Omega\rangle = \langle \Omega, T\Omega\rangle$ we get

$$T_{n,m} = c\cdot\langle a^*(f_1)\cdot\ldots\cdot a^*(f_n)\Omega, a^*(g_1)\cdot\ldots\cdot a^*(g_m)\Omega\rangle$$

in all cases, $n, m \in \mathbf{N}$. Trivially $T_{n,0} = T_{0,n} = \delta_n\cdot c$, so for any $n, m \in \mathbf{N}\cup\{0\}$ we have

$$\langle f_1\wedge\cdots\wedge f_n, (T - c\cdot I)g_1\wedge\cdots\wedge g_m\rangle = 0$$

where $f_1\wedge\cdots\wedge f_n$ and $g_1\wedge\cdots\wedge g_m$ means Ω for n and m zero respectively. Hence $T = c\cdot I$ on all of $\mathcal{H}$, since product vectors span $\mathcal{F}_\wedge(\mathcal{H})$. Then doing to a well-known theorem (see for example [B-R 1, p.47]) it follows that the representation is irreducible, proving the claim. ∎

The CAR-algebra has an equivalent formulation, in term of the *Clifford algebra*, which is the analogous description to that of the *CCR-algebra* (see chapter 2).

Define $\pi(f)$ by $\pi(f) = \frac{1}{\sqrt{2}}(a(f) + a(f)^*)$. Then $a(f)$ and $a(f)^*$ can be recovered from the $\pi(f)$ by the formula $a(f) = \frac{1}{\sqrt{2}}(\pi(f) + i\pi(if))$ and $a(f)^* = \frac{1}{\sqrt{2}}(\pi(f) - i\pi(if))$. Moreover, it follows from the CAR that

$$\begin{aligned}
[\pi(f), \pi(g)]_+ &= \frac{1}{2}(\, [a(f), a(g)]_+ + [a(f), a(g)^*]_+ \\
&\quad + [a(f)^*, a(g)]_+ + [a(f)^*, a(g)^*]_+) \\
&= \frac{1}{2}(\langle f, g\rangle + \overline{\langle f, g\rangle})\cdot I = Re\langle f, g\rangle \cdot I \\
&= \tau(f, g)\cdot I
\end{aligned}$$

where $\tau(f,g) = Re\langle f, g\rangle$ is a real positive symmetric bilinear form on $\mathcal{H}$. Conversely, the CAR can be recovered from the relations $[\pi(f), \pi(g)]_+ = \tau(f,g)\cdot I$. *The orthogonal group* $\mathcal{O}(\mathcal{H})$ consists of those real linear invertible mappings $T : \mathcal{H} \to \mathcal{H}$ such that $\tau(Tf, Tg) = \tau(f,g)$, for all $f, g \in \mathcal{H}$. Consider now the complex Clifford algebra over $\mathcal{H}$, as a real Hilbert space. Then the above relations become $a(f) = \frac{1}{\sqrt{2}}(\pi(f) + i\pi(Jf))$ and $a(f)^* = \frac{1}{\sqrt{2}}(\pi(f) - i\pi(Jf))$, where we have introduced a complex structure on the real Hilbert space through the operator J (this is done in detail in the beginning of section 1.4). For each $T \in \mathcal{O}(\mathcal{H})$, define $\pi_T(f)$ by $\pi_T(f) = \pi(T^{-1}f)$, then is $[\pi_T(f), \pi_T(g)]_+ = \tau(f,g)\cdot I$. Thus the mapping $\pi(f) \to \pi_T(f)$ defines an automorphism of the Clifford algebra, and these automorphisms form an automorphism group. It follows (see section 1.4 below) that these orthogonal transformations have a splitting into a sum $T = U + V$, where U and V are Bogoliubov transformations (see formula (1.3) and compare with formula (1.5) and (1.6) in section 1.4) corresponding to the above mention equivalence of the CAR-algebra and the Clifford algebra.

At this point it is natural to ask the question: For which $T \in \mathcal{O}(\mathcal{H})$ is this automorphism unitary implementable in the Fock representation, i.e. for which $T \in \mathcal{O}(\mathcal{H})$ does there exists a unitary operator U_T on $\mathcal{F}_\wedge(\mathcal{H})$ such that $\pi_T(f) = U_T^{-1}\pi(f)U_T$ for all $f \in \mathcal{H}$? This question has been answered by Shale and Stinespring in [S-S] and is covered in detail by Araki in [Ar] and Lundberg in [Lu 2]. We return to this question in section 1.4. But first we will discuss the second quantization, and some of its generalizations.

1.3 The second quantization in $\mathcal{F}_\wedge(\mathcal{H})$.

In this section we will describe the second quantization, based on the explicit Fock Hilbert space construction in the anti-symmetric case, given in section 1.1, and on the Fock representation of the CAR-algebra, given in section 1.2. This section will be used frequently in the construction of a special representation of the *spin* algebra and is therefore essential for the rest of this paper.

The basis idea of this section goes back to V. Fock ([Fo]), in 1932. But it was J.M. Cook, suppervised by I. E. Segal, who has constructed the method called second quantization in details ([Co]), in 1953. The method gave a nice mathematical frame for handling many problems in quantum physics with infinitely many degrees of freedom. For anorther exelent, but somehow more general treatement of second quantization we refer to the book of F. A. Berezin, [Be], from 1966. The method of second quantization is, shortly stated, the method of lifting one-particle operators, on a Hilbert space, to many-particle operators, on the Fock Hilbert spaces, whenever it is possible (in the case of the CAR-algebra, one use the anti-symmetric Fock Hilbert space). The method of second quantization has been generalized by Araki in [Ar], Lundberg in [Lu 1] and [Lu 2] and others, in the sence that they lift trace-class operators and Hilbert-Schmidt operators, fulfilling some additional properties, to the CAR-algebra or to operators acting in the Fock Hilbert spaces. Our method in constructing the second quantization is somehow different from that given by Cook in [Co].

Let A be a skew-selfadjoint linear operator on $\mathcal{H}$. Suppose for a while that A is a bounded operator, then we may define the operators $U(A)_n$ acting on Ω and on product vectors in $\wedge^n\mathcal{H}$ (for arbitrary $n \in \mathbb{N}$) by $U(A)_0\Omega = \Omega$ and

$$U(A)_n(f_1 \wedge \cdots \wedge f_n) = e^A f_1 \wedge \cdots \wedge e^A f_n$$

Notice that

$$\begin{aligned} &\langle U(A)_n(f_1 \wedge \cdots \wedge f_n), U(A)_n(g_1 \wedge \cdots \wedge g_n)\rangle \\ &\quad = \det\{\left\langle e^A f_i, e^A g_j\right\rangle\}_{i,j=1,\ldots,n} \\ &\quad = \det\{\langle f_i, g_j\rangle\}_{i,j=1,\ldots,n} \end{aligned}$$

$$= \langle f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_n \rangle$$

so extension by linearity and continuity gives that each $U(A)_n$ is a well-defined unitary operator on $\wedge^n \mathcal{H}$. Then $U(tA)_n$ is a strongly continuous one-parameter unitary group (see for example [R-S 1, p. 265]) of operators on $\wedge^n \mathcal{H}$, $t \in \mathbf{R}$. Hence, by a transformation of Stone's theorem (see [R-S 1, p. 266]) to skew-selfadjoint operators, implies that there exists a skew-selfadjoint operator $dU(A)_n$ on $\wedge^n \mathcal{H}$ such that

$$U(tA)_n = e^{t \cdot dU(A)_n}$$

$t \in \mathbf{R}$. The closed densely defined operator $dU(A)_n$ is called the infinitesimal generator, or just the generator, of $U(A)_n$, since

$$dU(A)_n = \frac{d}{dt}\bigg|_{t=0} U(tA)_n$$

on its domain, consisting of those vectors $F_n \in \wedge^n \mathcal{H}$ such that the limit of $t^{-1}(U(tA)_n - I)F_n$ exists. Let $F_n = f_1 \wedge \cdots \wedge f_n$ be an arbitrary product vector in $\wedge^n \mathcal{H}$, then is

$$dU(A)_n F_n = \sum_{i=1}^{n} f_1 \wedge \cdots \wedge f_{i-1} \wedge Af_i \wedge f_{i+1} \wedge \cdots \wedge f_n \tag{1.4}$$

which is a well-defined finite linear combination of n-particle product vectors in $\wedge^n \mathcal{H}$. Observe now, that the algebraic direct sum $\mathcal{D} = \oplus_{alg} \wedge^n \mathcal{H}$ equals $\mathcal{D}_0 = \mathcal{A}_0 \Omega$, where $\mathcal{A}_0$ is the C^*-algebra generated by $\{a_0(f) : f \in \mathcal{H}\}$, which is the Fock space realization of the CAR-algebra. Since $U(tA)_0 \Omega = \Omega$ is independent of t, it follows that $dU(A)_0 \Omega = 0$. We may extend $dU(A)_n$ to $\mathcal{D}$. Hence $dU(A) = \oplus_{n=0}^{\infty} dU(A)_n$ is a well-defined skew-symmetric operator, with dense invariant domain $\mathcal{D}$. Now, consider a possible unbounded skew-selfadjoint operator A. We then define $dU(A)_n$ directly by (1.4) on product vectors in $\wedge^n \mathcal{H}$, such that each one-particle vector, in the product vector, belongs to $\mathcal{D}(A)$, we denote these vectors by $\mathcal{D}(A)^{\wedge n}$, for each $n \in \mathbf{N}$, and $dU(A)_0 \Omega = 0$. Put $dU(A) = \oplus_{n=0}^{\infty} dU(A)_n$ on $\oplus_{alg} \mathcal{D}(A)^{\wedge n}$. We will now show that $dU(A)$ has a dense set of analytic vectors (wrt. the norm topology, hence also wrt. the week topology), in the case of a general skew-selfadjoint operator A. First, a direct calculation gives

$$dU(A)(f_1 \wedge \cdots \wedge f_n) = (n!)^{\frac{1}{2}} P_{\wedge} dU(A)_{\otimes} (f_1 \otimes \cdots \otimes f_n)$$

for $f_1, \ldots, f_n \in \mathcal{D}(A)$, where

$$dU(A)_{\otimes}(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^{n} f_1 \otimes \cdots \otimes f_{i-1} \otimes Af_i \otimes f_{i+1} \otimes \cdots \otimes f_n$$

and in the same manner we get

$$dU(A)^k(f_1 \wedge \cdots \wedge f_n) = (n!)^{\frac{1}{2}} P_{\wedge} dU(A)^k_{\otimes}(f_1 \otimes \cdots \otimes f_n)$$

Now if each f_j, in the product vector, is an analytic vector for A, then there exists a $M_j < \infty$, for each f_j, such that $\|Af_j\| \leq M_j \|f_j\| < \infty$. The skew-selfajointness of A implies that $\mathcal{D}(A)$ has a dense set of analytic vectors (doing to a corollary of Nelson's analytic vector theorem, see for example [R-S 2, p. 202]). For product vectors of analytic vectors for A we have

$$\begin{aligned} &\|dU(A)_{\otimes}(f_1 \otimes \cdots \otimes f_n)\|^2 \\ &= \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \langle Af_i, f_i \rangle \langle f_j, Af_j \rangle \prod_{\substack{m=1 \\ m \neq i,j}}^{n} \langle f_m, f_m \rangle + \sum_{i=1}^{n} \langle Af_i, Af_i \rangle \prod_{\substack{m=1 \\ m \neq i}}^{n} \langle f_m, f_m \rangle \\ &\leq M^2 \sum_{i,j=1}^{n} \prod_{m=1}^{n} \|f_m\|^2 = n^2 \cdot M^2 \cdot \|f_1 \otimes \cdots \otimes f_n\|^2 \end{aligned}$$

where $M = \max\{M_1, \ldots, M_n\}$. Hence

$$\begin{aligned} \|dU(A)^k(f_1 \wedge \cdots \wedge f_n)\| &\leq (n!)^{\frac{1}{2}} \|dU(A)^k_{\otimes}(f_1 \otimes \cdots \otimes f_n)\| \\ &\leq (n!)^{\frac{1}{2}} n^k M^k \|f_1 \otimes \cdots \otimes f_n\| \end{aligned}$$

and then, for any n-particle product vector $f_1 \wedge \cdots \wedge f_n$ such that each f_j is an analytic vector for A, is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k!} \|dU(A)^k(f_1 \wedge \cdots \wedge f_n)\| &\leq \sum_{k=0}^{\infty} \frac{(n \cdot M)^k}{k!} \|f_1 \otimes \cdots \otimes f_n\| (n!)^{\frac{1}{2}} \\ &= (n!)^{\frac{1}{2}} e^{n \cdot M} \|f_1 \otimes \cdots \otimes f_n\| \\ &< \infty \end{aligned}$$

Which means that the set of finite linear combination of Ω and n-particle product vectors $f_1 \wedge \cdots \wedge f_n$, $n \in \mathbf{N}$, such that each vector f_j is an analytic vector for A, form a dense set of analytic vectors for $dU(A)$. We denote the set of all analytic vectors for $dU(A)$ by $\mathcal{D}_A$ (for an alternative proof, see [R-S 2, p. 205]). Notice that in case of A bounded the above M and all M_j may be chosen equal to $\|A\|$, so $\mathcal{D}_A$ becomes all of $\mathcal{D}$ (any vector for a bounded operator A is an analytic vector for A). Then a transformation of Nelson's theorem (see for example [R-S 2, p. 202]) to essentially skew-selfadjoint operators, state that the operator $dU(A)$ is essentially skew-selfadjoint (which can also be seen by a quite different argument given, for example, in [R-S 1, p. 302]). Hence the closure of $dU(A)$, which we also denote by $dU(A)$, is skew-selfadjoint, and generates a strongly continuous one-parameter unitary group, by a transformation of Stone's theorem to skew-selfadjoint operators. We denote this strongly continuous unitary one-parameter group by $U(tA)$.

The above mapping $A \to dU(A)$, mapping skew-selfadjoint operators on $\mathcal{H}$ into skew-selfadjoint operators on $\mathcal{F}_\wedge(\mathcal{H})$, is called *the second quantization* mapping.

Theorem 2 *The mapping of second quantization, $A \to dU(A)$, on skew-selfadjoint operators A in $\mathcal{H}$ fulfil*

1) $U(tA)a_0(f)U(-tA) = a_0(e^{tA}f)$, *for analytic vectors f for A and $t \in \mathbf{R}$.*

2) $\overline{[dU(A), a_0{}^*(f)]} = a_0{}^*(Af)$, *for all $f \in \mathcal{D}(A)$.*

3) $dU([A,B]) = \overline{[dU(A), dU(B)]}$, *at least for A and B bounded and skew-selfadjoint on $\mathcal{H}$.*

Proof. This theorem combines theorem 2 and 5 in [Co]. However we bring some alternative proofs.

1) On an arbitrary n-particle product vector $f_1 \wedge \cdots \wedge f_n \in \mathcal{D}_A$, such that each f_j is an analytic vector for A, we have

$$\begin{aligned} U(tA)a_0{}^*(f)U(tA)^*(f_1 \wedge \cdots \wedge f_n) &= U(tA)(f \wedge e^{-tA}f_1 \wedge \cdots \wedge e^{-tA}f_n) \\ &= e^{tA}f \wedge f_1 \wedge \cdots \wedge f_n \\ &= a_0{}^*(e^{tA}f)(f_1 \wedge \cdots \wedge f_n) \end{aligned}$$

hence, $U(tA)a_0{}^*(f)U(-tA) = a_0{}^*(e^{tA}f)$ on $\mathcal{F}_\wedge(\mathcal{H})$, and the adjoint relation

$$U(tA)a_0(f)U(-tA) = a_0(e^{tA}f)$$

on $\mathcal{F}_\wedge(\mathcal{H})$, for all skew-selfadjoint operators A on $\mathcal{H}$, all analytic vectors f for A in $\mathcal{H}$ and all $t \in \mathbf{R}$.

2) For each analytic vector f for A, it follows from 1) that

$$\begin{aligned}
[dU(A), a_0{}^*(f)] &= \frac{d}{dt}\Big|_{t=0} e^{t\cdot dU(A)} a_0{}^*(f) e^{-t\cdot dU(A)} \\
&= \frac{d}{dt}\Big|_{t=0} a_0{}^*(e^{tA} f) \\
&= a_0{}^*(Af)
\end{aligned}$$

on the domain $\mathcal{D}_A$ of $dU(A)$, giving the desired formula for all skew-selfadjoint operators A on $\mathcal{H}$ and for all analytic vectors f for A, dense in $\mathcal{D}(A)$, hence for all $f \in \mathcal{D}(A)$.

3) We give two different proofs for this part, where the first is the more elegant one. For $f \in \mathcal{H}$ and A and B bounded and skew-selfadjoint, is

$$\begin{aligned}
&[dU([A,B]), a_0{}^*(f)] \\
&= a_0{}^*([A,B]f) \\
&= [dU(A), a_0{}^*(Bf)] - [dU(B), a_0{}^*(Af)] \\
&= [dU(A), [dU(B), a_0{}^*(f)]] - [dU(B), [dU(A), a_0{}^*(f)]] \\
&= [dU(A), dU(B)]a_0{}^*(f) - a_0{}^*(f)[dU(A), dU(B)] \\
&= [[dU(A), dU(B)], a_0{}^*(f)]
\end{aligned}$$

on $\mathcal{D}$. Hence the irreducibility of the Fock representation of the CAR-algebra, by theorem 1, and the fact that Ω is canceled by $dU(\cdot)$ gives the desired formula. We now turn to the second proof of part 3), this proof doesn't only give us some information on the Lie algebra level, but also on the group level. Let $t, s \in \mathbf{R}$. Since

$$\begin{aligned}
e^{tA} e^{sB} e^{-tA} &= \sum_{n=0}^{\infty} \frac{s^n}{n!} e^{tA} B^n e^{-tA} \\
&= \sum_{n=0}^{\infty} \frac{s^n}{n!} (e^{tA} B e^{-tA})^n \\
&= e^{s\cdot(e^{tA} B e^{-tA})}
\end{aligned}$$

then we have

$$\begin{aligned}
&U(tA)U(sB)U(tA)^*a_0(f)(U(tA)U(sB)U(tA)^*)^* \\
&= U(tA)U(sB)a_0(e^{-tA}f)U(sB)^*U(tA)^* \\
&= a_0(e^{tA}e^{sB}e^{-tA}f) \\
&= a_0(e^{s\cdot(e^{tA}Be^{-tA})}f) \\
&= U(s\cdot e^{tA}Be^{-tA})a_0(f)U(s\cdot e^{tA}Be^{-tA})^*
\end{aligned}$$

Since the Fock representation of the CAR-algebra is irreduceble, by theorem 1, and $U(tA)U(sB)U(tA)^*$ and $U(e^{tA}s\cdot Be^{-tA})$ is unitary it follows that

$$U(tA)U(sB)U(tA)^* = c(t,s)\cdot U(e^{tA}sBe^{-tA})$$

where $|c(t,s)| = 1$. But, since $U(tC)\Omega = \Omega$, for any skew-selfadjoint operator C and all $t \in \mathbf{R}$, it follows that $c(t,s) = 1$. So, in fact, we have that $U(tA)U(sB)U(tA)^* = U(e^{tA}s\cdot Be^{-tA})$, and for any $F \in \mathcal{D}$ we then get

$$\begin{aligned}
&[dU(A), dU(B)]F \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} U(tA)U(sB)U(tA)^*F \\
&= \left.\frac{d}{dt}\right|_{t=0} U(tA)dU(B)U(tA)^*F \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} U(e^{tA}s\cdot Be^{-tA})F \\
&= \left.\frac{d}{dt}\right|_{t=0} U(e^{tA}Be^{-tA})F \\
&= dU([A,B])F
\end{aligned}$$

from which the desired formula follows, ones more. Alternatively we could easily prove 2) by direct calculation first on product vectors using equation (1.4), and then extending by linearity to all of $\mathcal{D}$. In fact we will use this method in case of the CCR-algebra which is treated in chapter 2 ∎

Later on, in section 1.4, we will use this result, but only for bounded operators A and B. Notice that theorem 2, part 3) gives that the mapping $A \longrightarrow dU(A)$ is a Lie algebra homomorphism. The second quantization is sometimes called the Fock-Cook quantization mapping or just the Cook quantization mapping.

A standard example arises for $A = iI$, whereby $dU(A)(f_1 \wedge \cdots \wedge f_n) = i \cdot n \cdot (f_1 \wedge \cdots \wedge f_n)$, we call $-idU(A)$ for the number operator N in $\mathcal{F}_\wedge(\mathcal{H})$, with dense domain $\mathcal{D}(N) = \{F = \oplus_{n=0}^{\infty} F_n \in \mathcal{F}_\wedge(\mathcal{H}) : \sum_{n=0}^{\infty} n^2 \cdot \|F_n\|^2 < \infty\}$. So $N(\oplus_{n=0}^{\infty} F_n) = \oplus_{n=0}^{\infty}(n \cdot F_n)$ for each $\oplus_{n=0}^{\infty} F_n \in \mathcal{D}(N)$. Since N is given in its spectral representation, it is evidently selfadjoint (by von Neumann's theorem, see for example [R-S 1, p. 275]), as it should be, because $dU(A)$ is skew-selfadjoint.

As a historical remark we notice that Cook only considered second quatization in the Fock representation, this was generalized by Araki and Wyss to C^*-algebras. In 1964 Araki and Wyss in [A-W] showed that $dU(A)$ belongs to the CAR-algebra if A is a trace-class, skew-selfadjoint operator on $\mathcal{H}$, and that $dU(A)$, for finite rank operators $A : f \rightarrow Af = \sum_{i=1}^{n} \alpha_i \langle e_i, f\rangle e_i$, is given by $dU(A) = \sum_{i=1}^{n} \alpha_i a(e_i)^* a(e_i)$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal set in $\mathcal{H}$ (most authors consider selfadjoint operators instead of skew-selfadjoint operators, we have translated their statements, as we have translated some theorems, such as Stone's and Nelson's theorem, used above). In [Lu 1], 1976, Lundberg extend the second quantization mapping, in a quasifree representation labelled by $T, 0 \leq T \leq I$, to all bounded skew-selfadjoint operators A such that $\mathrm{Tr}(T^{\frac{1}{2}} A(I - T) A T^{\frac{1}{2}}) < \infty$, with $dU(A)$ affiliated with the CAR-algebra. In [Ar], 1985, and [Lu 2], 1990, Araki and Lundberg generalized the consept of second quantization, in the sence that they lift operators in the restricted orthogonal group to skew-selfadjoint operators acting in the Fock Hilbert spaces. We turn to this discussion in the next section. This idea was first used for the restricted unitary group, by Lundberg in [Lu 1], 1976. We point out that the formalism in [Ar] and [Lu 2] is quite different. Finally, we will mention that Langmann recently has used the idea of quasi-free second quantization to obtain a current algebra in (3+1)-dimensional quantum field theory, with the well-known Mickelsson-Rajeev Schwinger term (see [Mi]), for futher details, see [La].

1.4 The infinite dimensional spin representation.

In this section we construct the socalled spin representation of the restricted orthogonal group on a Lie algebra level, by ideas similar to those of second quantization. This dicussion are closely related to the answer of the question, stated in the end of section 1.2, for which orthogonal operators T is the automorphism, given by $\pi(f) \to \pi_T(f) = \pi(T^{-1}f)$, of the Clifford algebra, unitary implementable (in the Fock representation). In this section we will use the former sections 1.1 - 1.3.

Many authors have studied the spin representation of which we only mention a few here, [Ar], [Lu 2], [P-S] and [S-S]. This section is based on these papers. However, after finishing the first part of this thesis, a very interesting book has come to our knowlegde, namely [V-Z] from 1990, especially the first article of A.M. Vershik, who deals with some of the same problems and in the same formalism as we do. But Vershik consider the infinite dimensional restricted orthogonal (which he calls metagonal, in analogy with the metaplectic case) Lie algebra and group as the inductive limits of the corresponding finite dimensional ones, in his treatment of the infinite dimensional spin group.

Any real linear mapping $T : \mathcal{H} \to \mathcal{H}$ can be split into a sum of a complex linear mappings T_1 and a complex anti-linear mapping T_2 as $T = T_1 + T_2$. If $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for the complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and we let u_k and v_k denote e_k and ie_k, respectively, then $\{u_k, v_k\}_{k\in\mathbf{N}}$ form an orthonormal system with respect to $\tau(\cdot,\cdot) = Re\,\langle\cdot,\cdot\rangle$. We call the real span of $\{u_k, v_k\}_{k\in\mathbf{N}}$ for *the real Hilbert space*, $\mathcal{H}_r$, of $\mathcal{H}$, with inner product $\tau(\cdot,\cdot)$. Notice that $\mathcal{H}_r$ and $\mathcal{H}$ represent the same set. The complex structure on $\mathcal{H}$, given by multiplication by the imaginary unit i is reflected in $\mathcal{H}_r$ by a (real) linear bounded operator J, given by $Ju_k = v_k$ and $Jv_k = -u_k$. Then $J^2 = -I$ and $\tau(Jf, g) = -\tau(f, Jg)$, for all $f, g \in \mathcal{H}_r$, i.e. $J^\tau = -J$, where J^τ denote the transpose of J relative to $\tau(\cdot,\cdot)$. We say that J introduce a *complex structure* in $\mathcal{H}_r$. Because of the unique correspondence between $\mathcal{H}$ and $\mathcal{H}_r$, given by $e_k \leftrightarrow u_k$ and $ie_k \leftrightarrow v_k$, we will not emphasize on which space the operators applies and therefore drop the r-index, unless confusion may arise.

Define $T_1 = \frac{1}{2}(T - JTJ)$ and $T_2 = \frac{1}{2}(T + JTJ)$, then direct computations gives that T_1 and J commute, $T_1J = JT_1$, and that T_2 and J anti-commute, $T_2J = -JT_2$, which gives a precise mathematical meaning to the statements that T_1 is complex linear and that T_2 is complex anti-linear. So we have constructed a complex linear operator T_1 and a complex anti-linear operator T_2 such that $T = T_1 + T_2$. The subscripts 1 and 2 will in the following refer to this splitting. Consider the orthogonal group $\mathcal{O}(\mathcal{H})$, defined in the end of section 1.2, and let $T \in \mathcal{O}(\mathcal{H})$, with the above splitting $T = T_1 + T_2$. The adjoint $T_1{}^*$ of T_1 is the usual adjoint of a complex linear operator given by $\langle f, T_1 g\rangle = \langle T_1{}^* f, g\rangle$, for all $f, g \in \mathcal{H}$. But the adjoint, which we also denote by an asterisk as superscript, $T_2{}^*$ of T_2 is given by $\langle f, T_2 g\rangle = \langle g, T_2{}^* f\rangle$, for all $f, g \in \mathcal{H}$, due to the fact that T_2 is a complex linear mapping $\mathcal{H} \to \mathcal{H}^*$, where $\mathcal{H}^*$ denote the conjugated Hilbert space of $\mathcal{H}$, for futher details see section 1.6 later on. We restate that the adjoint operations, denoted by the asterisk's, means different things, corresponding to the subscript of the operator it is applied to. Since $T^\tau T$ is the identity on $\mathcal{H}_r$, for any $T \in \mathcal{O}(\mathcal{H})$, we get $T^{-1} = T^\tau$. Since

$$
\begin{aligned}
\tau(T^\tau f, g) &= \tau(f, Tg) = \tau(f, T_1 g) + \tau(f, T_2 g) \\
&= \tau(T_1{}^* f, g) + \tau(g, T_2{}^* f) = \tau(T_1{}^* f, g) + \tau(T_2{}^* f, g) \\
&= \tau((T_1{}^* + T_2{}^*) f, g)
\end{aligned}
$$

for all $f, g \in \mathcal{H}$, where we have used the explicit form of $\tau(\cdot,\cdot)$ and its symmetry, it follows that

$$T^{-1} = T^\tau = T_1{}^* + T_2{}^*$$

So $T_1{}^* = (T^\tau)_1$ and $T_2{}^* = (T^\tau)_2$. Moreover

$$
\begin{aligned}
I &= T^{-1}T = T^\tau T = (T_1{}^* + T_2{}^*)(T_1 + T_2) \\
&= (T_1{}^* T_1 + T_2{}^* T_2) + (T_1{}^* T_2 + T_2{}^* T_1)
\end{aligned}
$$

so we get

$$T_1{}^* T_1 + T_2{}^* T_2 = I \tag{1.5}$$

and

$$T_1{}^* T_2 + T_2{}^* T_1 = 0 \tag{1.6}$$

since the left side of (1.5) is complex linear and the left side of (1.6) is complex anti-linear. Hence, from this and an analogous computation, it follows that T_1 and T_2 fulfil the criterion of being Bogoliubov transformations. For later use, we now define *the restricted orthogonal group* $\mathcal{O}_2(\mathcal{H})$ as the subgroup of $\mathcal{O}(\mathcal{H})$ given by

$$\mathcal{O}_2(\mathcal{H}) = \{T \in \mathcal{O}(\mathcal{H}) : T_2 \in \mathbf{L}_2(\mathcal{H})\}$$

where $\mathbf{L}_2(\mathcal{H})$ denotes the Hilbert-Schmidt operators on $\mathcal{H}$. From above it follows that $\mathcal{O}_2(\mathcal{H})$ is a group indeed. Some authors denote $\mathcal{O}_2(\mathcal{H})$ by $\mathcal{O}_J(\mathcal{H})$, due to the fact that it consists of the subgroup of $\mathcal{O}(\mathcal{H})$ such that $[T, J] = -2JT_2 \in \mathbf{L}_2(\mathcal{H})$, see for example [V-Z], who call the operators in $\mathcal{O}_2(\mathcal{H})$ for *almost linear operators.*

The group $\mathcal{O}_2(\mathcal{H})$ can be given the structure of a topological group in several different ways, which is typical for infinite dimensional groups. The strongest topology is given by the uniform topology on the complex linear part and the Hilbert-Schmidt topology on the complex anti-linear part. However, in some applications one has to use a weaker topology on the linear part, for example the strong topology (see [Ar] paragraf 6). In [Ar, p. 77] and in [P-S, p. 245] it is shown that in both these topologies, there are two connected components of $\mathcal{O}_2(\mathcal{H})$, each which is simply connected. For example, in the strongest topology it follows easily that

$$\|T^{-1}\|_{\mathcal{O}_2} = \|T_1{}^*\| + \|T_2{}^*\|_{HS} = \|T\|_{\mathcal{O}_2} < \infty$$

where $\|T\|_{\mathcal{O}_2} = \|T_1\| + \|T_2\|_{HS}$, by definition, that

$$\begin{aligned}\|ST\|_{\mathcal{O}_2} \leq\ & \|S_1\| \cdot \|T_1\| + \|S_2\|_{HS} \cdot \|T_2\|_{HS} \\ & + \|S_1\| \cdot \|T_2\|_{HS} + \|S_2\|_{HS} \cdot \|T_1\| < \infty\end{aligned}$$

since $\|\cdot\| \leq \|\cdot\|_{HS}$, where the *HS*-index on the norm symbol, means the Hilbert-Schmidt norm, that

$$\begin{aligned}\|ST - HK\|_{\mathcal{O}_2} \leq\ & \|S_1\| \cdot \|T_1 - K_1\| + \|S_1 - H_1\| \cdot \|K_1\| \\ & + \|S_2\|_{HS} \cdot \|T_2 - K_2\|_{HS} + \|S_2 - H_2\|_{HS} \cdot \|K_2\|_{HS} \\ & + \|S_1\| \cdot \|T_2 - K_2\|_{HS} + \|S_1 - H_1\| \cdot \|K_2\|_{HS} \\ & + \|S_2\|_{HS} \cdot \|T_1 - K_1\| + \|S_2 - H_2\|_{HS} \cdot \|K_1\|\end{aligned}$$

and that

$$\begin{aligned}\|T^{-1} - S^{-1}\|_{\mathcal{O}_2} &= \|T_1{}^* - S_1{}^*\| + \|T_2{}^* - S_2{}^*\|_{HS} \\ &= \|T_1 - S_1\| + \|T_2 - S_2\|_{HS} = \|T - S\|_{\mathcal{O}_2}\end{aligned}$$

proving that $\mathcal{O}_2(\mathcal{H})$ in fact is a topological group in the case of uniform topology on the linear part and Hilbert-Schmidt topology on the anti-linear part.

The choice of topology on $\mathcal{O}_2(\mathcal{H})$ determinates the Lie algebra of $\mathcal{O}_2(\mathcal{H})$. Our choice of "pre-Lie-algebra" $o_2(\mathcal{H})$ is

$$o_2(\mathcal{H}) = \{A \in \mathbf{L}_r(\mathcal{H}) : A^\tau = -A, A_2 \in \mathbf{L}_2(\mathcal{H})\}$$

where $\mathbf{L}_r(\mathcal{H})$ denote the real linear bounded operators on $\mathcal{H}$. The phrase "pre-Lie-algebra" means that we in some applications have to enlarge the "pre-Lie-algebra" to allow operators with unbounded linear part (for futher details see [Ar, p. 81 and 104]). The demand $A^\tau = -A$ means that $A_1{}^* = -A_1$ and $A_2{}^* = -A_2$, i.e. both the linear and the anti-linear part of A are skew-selfadjoint (in their respective senses). In what follows we shall in particular consider $\mathcal{O}_2(\mathcal{H})$ in a neighbourhood of the identity, generated from $o_2(\mathcal{H})$ by the exponential mapping. Notice that the exponential mapping from an infinite dimensional Lie algebra to the corresponding infinite dimesional Lie group, modelled on a general topological vector space, need not be locally ono-to-one nor locally onto, for example this is the case for $\exp : \mathit{Vect}(S^1) \to \mathit{Diff}(S^1)$, [P-S, p. 28]. In cases where the vector space is a Banach space there is a well-developed theory, which is quite parrallel to the theory of finite dimensional Lie groups ([P-S, p. 26]).

We shall now return, as promised, to the question stated in the end of section 1.2, for which $T \in \mathcal{O}(\mathcal{H})$ is the automorphism, defined by $\pi(f) \to \pi_T(f) = \pi(T^{-1}f)$, of the Clifford algebra, unitary implementable in the Fock representation. In fact this question has already been answered by Shale and Stinespring in [S-S], 1965, as stated in the following theorem.

Theorem 3 *A unitary operator $U(T)$, which implements the automorphism $\pi(f) \to \pi_T(f)$ exists if and only if $T \in \mathcal{O}_2(\mathcal{H})$. Moreover, the operator $U(T)$ is unique up to a phase of modulus one.*

Proof. A proof can be found in [S-S], we make, however, a construction of $U(T)$ below, in a neighbourhood of the identity in $\mathcal{O}_2(\mathcal{H})$. ∎

Because the unique correspondance between the a's and the π's the irreducibility of the C^*-algebra generated by $\{a_0(f) : f \in \mathcal{H}\}$ imply that of the C^*-algebra generated by $\{\pi(f) : f \in \mathcal{H}\}$ as well. But $U(TS)^{-1}\pi(f)U(TS) =$

$(U(T)U(S))^{-1}\pi(f)(U(T)U(S))$, so $U(T)U(S)U(TS)^{-1}$ equals a constant times the identity I. Hence

$$U(T)U(S) = c(T,S) \cdot U(TS)$$

where $c(T,S) \in \mathbf{C}$. Now, the unitarity of $U(\cdot)$ force $c(T,S)$ to be of modulus one

$$\begin{aligned} \langle f,g\rangle &= \langle U(T)U(S)f, U(T)U(S)g\rangle \\ &= |c(T,S)|^2 \cdot \langle U(TS)f, U(TS)g\rangle \\ &= |c(T,S)|^2 \cdot \langle f,g\rangle \end{aligned}$$

for all $f,g \in \mathcal{H}$, giving that $|c(T,S)| = 1$. This means that the mapping $T \to U(T)$ is a projective representation of the restricted orthogonal group $\mathcal{O}_2(\mathcal{H})$.

The group cocycle $c(T,S)$ depends on the choice of the arbitrary phase in $U(T)$. In [Lu 2], Lundberg gave an explicit formula for the cocycle $c(T,S)$, by choosing $U(T)$ such that $c(T,S)$ is smooth in such a way that $U(\cdot)$ lift one-parameter groups into one-parameter groups, for T and S close to the identity. We do this below, by giving a constructive proof of the if-part of the above theorem of Shale and Stinespring in the case of T in a neighbourhood of the identity in $\mathcal{O}_2(\mathcal{H})$ consisting of elements of the form $T = e^A$, with $A \in o_2(\mathcal{H})$. This is done by constructing the spin representation, on a Lie algebra level, that is we construct $U(e^{sA})$ for $A \in o_2(\mathcal{H})$ and $s \in \mathbf{R}$, by constructing its skew-selfadjoint generator $dU(A)$, hence $U(e^{sA})$ is given by $e^{s\cdot dU(A)}$.

Consider first the complex linear part A_1 of $A \in o_2(\mathcal{H})$, which is skew-selfadjoint. In this case $dU(A_1)$ and $U(e^{sA_1})$ is constructed by the method of Cook's second quantization given in section 1.3, where we denoted $U(e^{sA_1})$ by $U(s \cdot A_1)$. Hence $dU(A_1)$ is skew-selfadjoint on $\mathcal{D}$, where $\mathcal{D}$ is the dense set of analytic vectors for $dU(A_1)$, given by vectors $F = \oplus_{n=0}^{\infty} F_n \in \mathcal{F}_\wedge(\mathcal{H})$ such that only finitely many F_n are non-zero, due to the boundedness of A and then of A_1. Observe that $dU(A_1) : \wedge^n\mathcal{H} \to \wedge^n\mathcal{H}$.

We now turn to the anti-linear part A_2 of $A \in o_2(\mathcal{H})$. The following construction appears in [Lu 2, p. 6], however, it is not discussed in detail. Since A_2 is a Hilbert-Schmidt operator, there exist two orthogonal sets $\{u_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ in $\mathcal{H}$, both spanning the range of A_2, such that A_2 has the representation

$$A_2 f = \sum_{i\in I} \langle f, v_i\rangle u_i$$

for any $f \in \mathcal{H}$. Notice that A_2 given on this form clearly is anti-linear, due to our convention that the inner product is anti-linear in the first argument. A direct calculation, using the definition of the adjoint of an anti-linear operator and the skew-selfadjointness of A_2, gives that

$$\sum_{i\in I} \langle f, v_i\rangle u_i = -\sum_{i\in I} \langle f, u_i\rangle v_i$$

Moreover $\|A_2\|_{HS}^2 = \sum_{i\in\mathbb{N}} \|A_2 e_i\|^2 = \sum_{i\in I} \|v_i\|^2 \cdot \|u_i\|^2$. This means that we may identify A_2 with a vector $\mathcal{A}_2 \in \wedge^2\mathcal{H}$, where

$$\mathcal{A}_2 = \sum_{i\in I} v_i \wedge u_i = \sqrt{2} \cdot \sum_{i\in I} v_i \otimes u_i$$

Observe that the mapping $A_2 \mapsto \mathcal{A}_2$ is not an isometry, but the mapping $A_2 \mapsto \frac{1}{\sqrt{2}}\mathcal{A}_2$ is. We are now able to map A_2 into an operator on $\mathcal{F}_\wedge(\mathcal{H})$ by generalizing the idea of the creation operator. For the rest of this chapter, we discard the 0-index on $a_0(f)$, since we shall only consider the Fock representation. Define the operator $a^*(A_2)$ by

$$a^*(A_2)\Omega = \mathcal{A}_2 \tag{1.7}$$

and

$$a^*(A_2)(f_1 \wedge \cdots \wedge f_n) = \mathcal{A}_2 \wedge f_1 \wedge \cdots \wedge f_n$$

and extend by linearity to the domain $\mathcal{D}$. Notice that the vector $\mathcal{A}_2 \wedge f_1 \wedge \cdots \wedge f_n$ is well-defined in $\wedge^{n+2}\mathcal{H}$ whenever $\mathcal{A}_2 \in \wedge^2\mathcal{H}$ takes the form of a finite linear combination of simple product vectors, $\mathcal{A}_2 = \sum_{i=1}^N g_{1,i} \wedge g_{2,i} \in \wedge^2\mathcal{H}$, where $N \in \mathbb{N}$. Any vector $\mathcal{A}_2 \in \wedge^2\mathcal{H}$ can be approximate by finite linear combination of such product vectors. Let $\mathcal{A}_2$ be an arbitrary vector in $\wedge^2\mathcal{H}$ and $\mathcal{A}_2^{(N)} = \sum_{i=1}^N g_{1,i} \wedge g_{2,i}$ an approximating sequence, i.e. $\mathcal{A}_2 = \lim_{N\to\infty} \mathcal{A}_2^{(N)}$, in $\wedge^2\mathcal{H}$. Then we may define $\mathcal{A}_2 \wedge f_1 \wedge \cdots \wedge f_n$ as the limit of $\mathcal{A}_2^{(N)} \wedge f_1 \wedge \cdots \wedge f_n$, in $\wedge^{n+2}\mathcal{H}$, as N tends to infinity. The limit depends only of the vector $\mathcal{A}_2$ (and ofcause of $f_1 \wedge \cdots \wedge f_n$), but not on the choise of approximating sequence $\mathcal{A}_2^{(N)}$, since $\|\mathcal{A}_2 \wedge f_1 \wedge \cdots \wedge f_n\| \leq \sqrt{(n+2)(n+1)} \cdot \|\mathcal{A}_2\| \cdot \|f_1 \wedge \cdots \wedge f_n\|$ as follows from below. Notice that

$a^*(A_2)$ is in general unbounded, since if $\{e_{i_1}, \ldots, e_{i_n}\}_{i_1<\cdots<i_n}$ is an orthonormal system orthogonal to $span\{v_i\}_{i\in I}$, which equals the range of A_2, then

$$\begin{aligned}\|a^*(A_2)(e_{i_1} \wedge \cdots \wedge e_{i_n})\| &= \|\mathcal{A}_2 \wedge e_{i_1} \wedge \cdots \wedge e_{i_n}\| \\ &= \sqrt{(n+2)(n+1)} \cdot \|\mathcal{A}_2\|\end{aligned}$$

which clearly is unbounded, due to the n-dependent factor. In fact the formula

$$\|a^*(A_2)F\| \leq \sqrt{(n+2)(n+1)} \cdot \|\mathcal{A}_2\| \cdot \|F\|$$

holds on any n-particle vector $F = F_n \in \wedge^n \mathcal{H}$, which is most easily obtained first on product basis vectors, hence by continuity on all $\wedge^n\mathcal{H}$. Hence the formula holds for any $F \in \mathcal{D}$, since if $F_n = 0$ for $n > N$ then each term $\sqrt{(n+2)(n+1)}$ is dominated by $\sqrt{(N+2)(N+1)}$ and we get the same formula but with N instead of n. So $a^*(A_2)$ is well-defined on $\mathcal{D}$, which we may choose as the domain. The domain $\mathcal{D}$ is evidently invariant under the action of $a^*(A_2)$, so $\mathcal{D}$ form a invariant dense set for $a^*(A_2)$. In fact $\mathcal{D}$ is a dense set of analytic vectors for $a^*(A_2)$, since

$$\|a^*(A_2)^k F\| \leq \sqrt{\frac{(N+2k)!}{N!}} \cdot \|\mathcal{A}_2\|^k \cdot \|F\|$$

for any $F \in \mathcal{D}$ such that $F_n = 0$ for $n > N$, so

$$\sum_{k=0}^{\infty} \frac{|t|^k}{k!} \|a^*(A_2)^k F\| \leq \sum_{k=0}^{\infty} \frac{c(k)}{k!} (|t|\,\|\mathcal{A}_2\|)^k \cdot \|F\|$$

which is finite for $|t| < (2\|\mathcal{A}_2\|)^{-1}$ and $F \in \mathcal{D}$, with $F_n = 0$ for $n > N$, by use of the ratio test, where $c(k) = \sqrt{\frac{(N+2k)!}{N!}}$.

Notice that $A_2 f = \sum_{i\in I} \langle f, v_i \rangle u_i = \frac{1}{2} \sum_{i \in I} a(f)(v_i \wedge u_i) = \frac{1}{2} a(f)\mathcal{A}_2 = \frac{1}{2} a(f) a^*(A_2)\Omega$, so $a^*(A_2 f)\Omega = \frac{1}{2}[a(f), a^*(A_2)]\Omega$, for all $f \in \mathcal{H}$. In fact this commutation relation holds, not only on Ω, but on all of $\mathcal{D}$. Let $F_n = f_1 \wedge \cdots \wedge f_n$, then

$$\begin{aligned}a(f)a^*(A_2)F_n &= a(f)(\mathcal{A}_2 \wedge F_n) \\ &= (a(f)\mathcal{A}_2) \wedge F_n + \mathcal{A}_2 \wedge (a(f)F_n) \\ &= (2a^*(A_2 f) + a^*(A_2)a(f))F_n\end{aligned}$$

where we have used the invariance of $\mathcal{D}$ under the action of $a(f)$, hence

$$\frac{1}{2}[a(f), a^*(A_2)] = a^*(A_2 f) \tag{1.8}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$. Observe that $a^*(A_2) : \wedge^n \mathcal{H} \to \wedge^{n+2}\mathcal{H}$. By $a(A_2)$ we denote the formal adjoint $a^*(A_2)^*$ of $a^*(A_2)$. It follows that it is well-defined on $\mathcal{D}$. Since $\langle \Omega, a^*(A_2)F\rangle = 0$, for all $F \in \mathcal{D}$, we have that $\Omega \in \mathcal{D}(a(A_2))$ and $a(A_2)\Omega = 0$. On $\wedge^1\mathcal{H}$ it follows analogous that $f \in \mathcal{D}(a(A_2))$ and $a(A_2)f = 0$, for each $f \in \mathcal{H}$. We continue by induction after $n \in \mathbf{N}$ using the fact that $a^*(A_2)$ increase the number of particle by two and using formula (1.8). Suppose that we have proved that $f_2 \wedge \cdots \wedge f_n \in \mathcal{D}(a(A_2))$ for any $f_2, \ldots, f_n \in \mathcal{H}$, then, for any $f_1 \in \mathcal{H}$ and $F \in \wedge^{n-2}\mathcal{H}$, is

$$\begin{aligned}
\langle f_1 \wedge \cdots \wedge f_n, a^*(A_2)F\rangle &= \langle a^*(f_1)(f_2 \wedge \cdots \wedge f_n), a^*(A_2)F\rangle \\
&= \langle f_2 \wedge \cdots \wedge f_n, a(f_1)a^*(A_2)F\rangle \\
&= \langle f_2 \wedge \cdots \wedge f_n, (a^*(A_2)a(f_1) + 2a^*(A_2 f_1))F\rangle \\
&= \langle (a^*(f_1)a(A_2) + 2a(A_2 f_1))(f_2 \wedge \cdots \wedge f_n), F\rangle
\end{aligned}$$

which is well-defined, since $f_2 \wedge \cdots \wedge f_n \in \mathcal{D}(a(A_2))$, i.e. $f_1 \wedge \cdots \wedge f_n \in \mathcal{D}(a(A_2))$ and $a(A_2)(f_1 \wedge \cdots \wedge f_n) = (a^*(f_1)a(A_2) + 2a(A_2 f_1))(f_2 \wedge \cdots \wedge f_n)$. Hence by linearity, it follows that $\mathcal{D} \subset \mathcal{D}(a(A_2))$. Let now $G = \oplus_{n=0}^{\infty} G_n \in \mathcal{D}$, such that $G_n = 0$, for $n > N$, and let $F = \oplus_{n=0}^{\infty} F_n \in \mathcal{D}$. Then is

$$|\langle G, a^*(A_2)F\rangle| \leq \sqrt{N(N-1)} \cdot \|\mathcal{A}_2\| \cdot \|G\| \cdot \|F\|$$

where we have used the estimate $\|a^*(A_2)F_n\| \leq \sqrt{(n+2)(n+1)}\|\mathcal{A}_2\| \cdot \|F_n\|$, for $n \leq N-2$. So

$$\|a(A_2)G\| = \sup_{F \in \mathcal{D}, \|F\|=1} |\langle a(A_2)G, F\rangle| \leq \sqrt{N(N-1)} \cdot \|\mathcal{A}_2\| \cdot \|G\|$$

since $\mathcal{D}$ is dense in $\mathcal{H}$. Moreover, we get

$$\|a(A_2)^k G\| \leq C(k) \cdot \|\mathcal{A}_2\|^k \cdot \|G\|$$

where $C(k) = 0$, for $2k > N$, and $C(k) = \sqrt{\frac{N!}{(N-2k)!}}$, for $2k \leq N$. Then

$$\sum_{k=0}^{\infty} \frac{|t|^k}{k!} \|a(A_2)^k G\| \leq \sum_{k=0}^{\infty} \frac{(|t| \cdot \|\mathcal{A}_2\|)^k}{k!} \cdot C(k) \cdot \|G\|$$

is finite, for $|t| < (2 \cdot \|\mathcal{A}_2\|)^{-1}$, due to the ratio test, since the quotient of the k'te and the $(k-1)$'te term converge to $2 \cdot |t| \cdot \|\mathcal{A}_2\|$, as $k \to \infty$, which is stricly less that one, for $|t| < (2 \cdot \|\mathcal{A}_2\|)^{-1}$. Hence each vector in $\mathcal{D}$ is an analytic vector for $a(A_2)$. Observe that $a(A_2) : \wedge^n \mathcal{H} \to \wedge^{n-2}\mathcal{H}$, yielding zero for $n = 0, 1$. Since $\mathcal{D}$ is invariant under the action of $a^*(f)$, the adjoint of formula (1.8) also holds on $\mathcal{D}$, $\frac{1}{2}[a(A_2), a^*(f)] = a(A_2 f)$, for all $f \in \mathcal{H}$. So $\mathcal{D}$ is invariant under the action of both $a^*(A_2)$ and $a(A_2)$.

Define $dU(A_2) = \frac{1}{2}(a(A_2) - a^*(A_2))$ on $\mathcal{D}$, for A_2 a skew-selfadjoint anti-linear Hilbert-Schmidt operator. Notice that $dU(A_2) : \wedge^n\mathcal{H} \to \wedge^{n-2}\mathcal{H} \oplus \wedge^{n+2}\mathcal{H}$. Define $dU(A) = dU(A_1) + dU(A_2)$ on $\mathcal{D}$, for $A = A_1 + A_2 \in o_2(\mathcal{H})$. Observe that $dU(A) : \wedge^n\mathcal{H} \to \wedge^{n-2}\mathcal{H} \oplus \wedge^n\mathcal{H} \oplus \wedge^{n+2}\mathcal{H}$.

Theorem 4 *The operator $dU(A)$ is essentially skew-selfadjoint and fulfils*

$$[dU(A), \pi(f)] = \pi(Af) \tag{1.9}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$ and all $A \in o_2(\mathcal{H})$,

$$\langle \Omega, dU(A)\Omega\rangle = 0 \tag{1.10}$$

for all $A \in o_2(\mathcal{H})$, and

$$\langle \Omega, dU(A)dU(B)\,\Omega\rangle = -\frac{1}{4}\langle \mathcal{A}_2, \mathcal{B}_2\rangle = \frac{1}{2}\mathrm{Tr}(B_2 A_2) \tag{1.11}$$

Proof. This theorem also appears in [Lu 2, theorem 2]. We elaborate the proof, since it is only sketched very briefly there. Since $dU(A)$ is skew-symmetric and has $\mathcal{D}$ as a dense set of analytic vectors, it follows by, a modification of, Nelson's theorem ([R-S 2, p. 202]), to skew-symmetric operators that $dU(A)$ is essential skew-selfadjoint. From theorem 2 we have that $[dU(A_1), \pi(f)] = \pi(A_1 f)$, on $\mathcal{D}$ for all $f \in \mathcal{H}$ (since it holds for π replaced by a and a^*, respectively). For the anti-linear part of A we get

$$\begin{aligned}
[dU(A_2), \pi(f)] &= \frac{1}{2}[a(A_2), \pi(f)] - \frac{1}{2}[a^*(A_2), \pi(f)] \\
&= 2^{-3/2}[a(A_2)a(f)^*] + 2^{-3/2}[a(A_2), a(f)] \\
&\quad - 2^{-3/2}[a^*(A_2), a(f)^*] - 2^{-3/2}[a^*(A_2), a(f)] \\
&= 2^{-1/2}a(A_2 f) + 0 - 0 + 2^{-1/2}a^*(A_2 f) \\
&= \pi(A_2 f)
\end{aligned}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$, where we have used that $\pi(f) = 2^{-1/2}(a(f) + a(f)^*)$, formula (1.8) and its adjoint, and that $[a^*(A_2), a(f)^*] = 0$ together with the adjoint relation on $\mathcal{D}$. Hence

$$\begin{aligned}
[dU(A), \pi(f)] &= [dU(A_1), \pi(f)] + [dU(A_2), \pi(f)] \\
&= \pi(A_1 f) + \pi(A_2 f) = \pi(Af)
\end{aligned}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$, proving (1.9). Moreover, since $dU(A_1)\Omega = 0$ and $a(A_2)\Omega = 0$, it follows that

$$\langle \Omega, dU(A)\Omega \rangle = -\frac{1}{2}\langle \Omega, a^*(A_2)\Omega \rangle = 0$$

proving (1.10). Finally

$$\begin{aligned}
\langle \Omega, dU(A)dU(B)\,\Omega \rangle &= -\langle dU(A)\Omega, dU(B)\Omega \rangle \\
&= -\langle \frac{1}{2}a^*(A_2)\Omega, \frac{1}{2}a^*(B_2)\Omega \rangle \\
&= -\frac{1}{4}\langle \mathcal{A}_2, \mathcal{B}_2 \rangle \\
&= \frac{1}{2}\mathrm{Tr}(B_2 A_2)
\end{aligned}$$

on $\mathcal{D}$, where we have used the anti-symmetry of $dU(A)$, definition (1.7) and in the last equality, the spectral forms of A_2 and B_2, respectively, together with the definition of the trace, proving (1.11). This will be explicitely done in the symmetric case (see theorem 17). ∎

We now define the unitary one-parameter group $U(e^{sA})$ by

$$U(e^{sA}) = e^{s \cdot dU(A)}$$

At this point we are rather close to have proved the if part of theorem 3 in a neighbourhood of the identity, but the fact that $U(e^{sA})$ create infinite many particles, is the reason why we first have to prove the following technical lemma, which doesn't seem to appear anywhere else.

Lemma 5 *Let G be an essential skew-selfadjoint operator with $\mathcal{D}$ as a dense set of analytical vectors and let B be any bounded operator leaving $\mathcal{D}$ invariant, both defined on the same Hilbert space. Then*

$$e^{sG}Be^{-sG} = \sum_{n=0}^{\infty} \frac{s^n}{n!}[G,B]^{(n)}$$

on $\mathcal{D}$, where $[G,B]^{(0)} = B$, $[G,B]^{(1)} = [G,B]$ and $[G,B]^{(n)} = [G,[G,B]^{(n-1)}]$, inductively, for $n \in \mathbf{N}$.

Proof. Let $f \in \mathcal{D}$, then is $e^{-sG}f$ well-defined, and so is $Be^{-sG}f$. Let $g \in \mathcal{D}$, then is $\langle e^{-sG}g, Be^{-sG}f\rangle$ well-defined, and induction, after $n \in \mathbf{N}$, gives

$$\frac{d^n}{ds^n}\langle e^{-sG}g, Be^{-sG}f\rangle = \langle e^{-sG}g, [G,B]^{(n)}e^{-sG}f\rangle$$

so, by Taylors formula, we get

$$\begin{aligned}\langle e^{-sG}g, Be^{-sG}f\rangle &= \sum_{n=0}^{\infty} \frac{s^n}{n!} \left.\frac{d^n}{ds^n}\right|_{s=0} \langle e^{-sG}g, Be^{-sG}f\rangle \\ &= \langle g, \sum_{n=0}^{\infty} \frac{s^n}{n!}[G,B]^{(n)}f\rangle\end{aligned}$$

for all $f,g \in \mathcal{D}$, which is dense in the Hilbert space. Then is $Be^{-sG}f \in \mathcal{D}((e^{-sG})^*)$ and $e^{sG}Be^{-sG} = \sum_{n=0}^{\infty} \frac{s^n}{n!}[G,B]^{(n)}$ on $\mathcal{D}$. ∎

The above lemma, with $G = dU(A)$ and $B = \pi(f)$ (and the substitution $s \to -s$), gives

$$\begin{aligned}U(e^{sA})^*\pi(f)U(e^{sA}) &= \sum_{n=0}^{\infty} \frac{(-s)^n}{n!}[dU(A), \pi(f)]^{(n)} \\ &= \sum_{n=0}^{\infty} \frac{(-s)^n}{n!}\pi(A^n f) \\ &= \pi(e^{-sA}f)\end{aligned}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$, where we have used (1.9). Hence we have the desired formula

$$U(T)^{-1}\pi(f)U(T) = \pi(T^{-1}f) = \pi_T(f) \tag{1.12}$$

on $\mathcal{H}$ for all $T = e^{sA}$, $A \in o_2(\mathcal{H})$, $f \in \mathcal{H}$ and $s \in \mathbf{R}$, where $U(T)$ has been explicit constructed, such that the arbitrary phase of $U(T)$ has been fixed on all one-parameter subgroups of $\mathcal{O}_2(\mathcal{H})$ of the form $T = e^{sA}$, $A \in o_2(\mathcal{H})$. We call $U : T \to U(T)$ *the spin representation* of the restricted orthogonal group and we define the *spin group* $Spin_2$ to be the group of all the unitary implementers $U(T)$, $T \in \mathcal{O}_2(\mathcal{H})$ from Shale and Stinespring's theorem (theorem 3). This construction is more transparent on the Lie algebra level.

Theorem 6 *The elements $dU(A)$, $A \in o_2(\mathcal{H})$ form a Lie algebra on $\mathcal{D}$ with bracket*

$$[dU(A), dU(B)] = dU([A, B]) + \omega(A, B) \cdot I \tag{1.13}$$

and the Lie algebra cocycle is given by

$$\omega(A, B) = -\frac{1}{2}\mathrm{Tr}([A_2, B_2]) = -\frac{i}{2}Im\langle \mathcal{A}_2, \mathcal{B}_2\rangle \tag{1.14}$$

This infinite dimensional Lie algebra is denoted the spin Lie algebra, $spin_2(\mathcal{H})$.

Proof. By use of (1.9) we have

$$\begin{aligned} [dU(A)dU(B), \pi(f)] &= dU(A)[dU(B), \pi(f)] + [dU(A), \pi(f)]dU(B) \\ &= dU(A)\pi(Bf) + \pi(Af)dU(B) \end{aligned}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$. So

$$\begin{aligned} [[dU(A), dU(B)], \pi(f)] &= dU(A)\pi(Bf) + \pi(Af)dU(B) \\ &\quad - dU(B)\pi(Af) - \pi(Bf)dU(A) \\ &= [dU(A), \pi(Bf)] + [\pi(Af), dU(B)] \\ &= \pi(ABf) - \pi(BAf) \\ &= \pi([A, B]f) \\ &= [dU([A, B]), \pi(f)] \end{aligned}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$, again by (1.9). Then the irreducibility of $\{\pi(f) : f \in \mathcal{H}\}$ gives

$$[dU(A), dU(B)] = dU([A, B]) + \omega(A, B) \cdot I$$

The relation (1.10) and (1.11) gives us an explicit formula for the Lie algebra cocycle, as follows

$$\begin{aligned}
\omega(A,B) &= \langle \Omega, \omega(A,B)\cdot I\,\Omega\rangle \\
&= \langle \Omega, [dU(A), dU(B)]\Omega\rangle \\
&= -\frac{1}{2}\mathrm{Tr}([A_2, B_2]) \\
&= -\frac{i}{2}Im\langle \mathcal{A}_2, \mathcal{B}_2\rangle
\end{aligned}$$

where we have used (1.10),(1.11), and (1.13). ∎

Notice that the trace of a commutator of two anti-linear Hilbert-Schmidt operators do not vanish, in general, as in the complex linear case. Moreover, it follows directly from (1.13) that ω is skew-symmetric and fulfils the Hochshild condition

$$\omega(AB,C) + \omega(BC,A) + \omega(CA,B) = 0$$

for all $A, B, C \in o_2(\mathcal{H})$, since $[AB,C]+[BC,A]+[CA,B] = 0$ for all bounded operators A, B and C. Then also the Jacobi identity is fulfilled

$$\omega([A,B],C) + \omega([B,C],A) + \omega([C,A],B) = 0$$

for all $A, B, C \in o_2(\mathcal{H})$, hence ω is a closed two-form.

Corollary 7 *The mapping $A \longrightarrow dU(A)$, $A \in o_2(\mathcal{H})$, is a projective representation, from the Lie algebra $o_2(\mathcal{H})$ onto the spin Lie algebra $spin_2(\mathcal{H})$, with cocycle given by* (1.14).

Proof. It follows directly from theorem 6. ∎

This cocycle is studied in details in [V-Z], article 1, paragraph 1.2, and it defines a non-trivial central extendsion of the Lie algebra $o_2(\mathcal{H})$. In the special case, when the linear part of A and B are trace-class operators, we are able to transform the cocycle term away, by a change of phase, as follows. Put $dU_0(A) = dU(A) - \frac{1}{2}\cdot\mathrm{Tr}(A_1)\cdot I$, then a straight forward calculation gives $[dU_0(A), dU_0(B)] = dU_0([A,B])$, by (1.14) and the fact that $\frac{1}{2}\cdot\mathrm{Tr}([A,B]_1) =$

$-\omega(A,B)$. So we put $U_0(e^{sA}) = e^{s\cdot dU_0(A)} = e^{-\frac{1}{2}s\mathrm{Tr}(A_1)}U(e^{sA})$, for $s \in \mathbf{R}$ close to zero. Then is $U(e^{sC}) = e^{\frac{1}{2}s(\mathrm{Tr}(C_1)-\mathrm{Tr}(A_1)-\mathrm{Tr}(B_1))}U(e^{sA})U(e^{sB})$, where C is given explicitly by the Campbell-Baker-Hausdorff formula, such that $e^{sC} = e^{sA}e^{sB}$, for s close to zero, i.e. the group cocycle $c(e^{sA}, e^{sB})$ is given by

$$c(e^{sA}, e^{sB}) = (\det(e^{sA_1}e^{sB_1}e^{-sC_1}))^{\frac{1}{2}}$$

for s close to zero, where $\det(e^D) = e^{\mathrm{Tr}(D)}$, for trace class operators D. This formula is also discussed in [P-S, p.88].

We end up this section by calculating an explicit formula for *the vacuum functional*, given by $c(s) = \langle \Omega, U(e^{sA})\Omega\rangle$, for $A \in o_2(\mathcal{H})$ and s in a neighbourhood of zero. The following construction was first described in [Lu 2, p.8], but with very few details. We bring the details below. Notice that $c(s)$ is analytic at $s = 0$, since Ω is an analytic vector for the generator $dU(A)$.

Put $T = e^{sA}$ and consider $\Omega_s = U(e^{sA})\Omega$. Then, by (1.12), we have $(a(T_1 f) + a^*(T_2 f))\Omega_s = U(T)a(f)\Omega = 0$, for all $f \in \mathcal{H}$. Define the anti-linear Hilbert-Schmidt operator K by $K = T_2T_1^{-1}$, where $T = T_1 + T_2$, T_2 is an anti-linear Hilbert-Schmidt operator and T_1 is linear and invertible for s sufficient small, by the Neumann serie. Observe that K is skew-selfadjoint $\langle f, Kg\rangle = \langle f, T_2T_1^{-1}g\rangle = \langle T_1^{-1}g, T_2{}^*f\rangle = \langle g, (T_1^{-1})^*T_2{}^*f\rangle = \langle g, -T_2T_1^{-1}f\rangle$ $= \langle g, -Kf\rangle$, for all $f, g \in \mathcal{H}$, where we have used that $(T_1^{-1})^*T_2{}^* = -T_2T_1^{-1}$, by formula (1.6) applied to $S = T_1^{-1} + T_2{}^*$. Then it follows from above that $(a(g) + a^*(Kg))\Omega_s = 0$, for all $g \in \mathcal{H}$ $(f = T_1^{-1}g)$.

Now we prove that

$$\Omega_s = c(s) \cdot e^{-\frac{1}{2}a^*(K)}\Omega$$

Put $\Omega_s = \oplus_{n=0}^{\infty}\Omega_n(s)$, where $\Omega_n(s) \in \wedge^n\mathcal{H}$. Note that $\Omega_0(s) = c(s)\cdot\Omega$. It then follows from the above that

$$\begin{aligned} 0 &= (a(g) + a^*(Kg))\Omega_s \\ &= a(g)\Omega_1(s) \oplus (\oplus_{n=1}^{\infty}(a(g)\Omega_{n+1}(s) + a^*(Kg)\Omega_{n-1}(s))) \end{aligned}$$

So $\Omega_1(s) = 0$, since $a(g)\Omega_1(s) = 0$, for all $g \in \mathcal{H}$, then by induction, it follows that $\Omega_{2n-1}(s) = 0$, for $n \in \mathbf{N}$. Moreover, since $a(g)\Omega_0(s) = 0$ and formula (1.8) holds, it follows that $a(g)(\Omega_2(s) + \frac{1}{2}a^*(K)\Omega_0(s)) = a(g)\Omega_2(s) + a^*(Kg)\Omega_0(s) = 0$, for all $g \in \mathcal{H}$, hence $\Omega_2(s) = -\frac{1}{2}a^*(K)\Omega_0(s) = -\frac{1}{2}c(s)\mathcal{K}$. Now, induction after $n \in \mathbf{N}$ gives that $\Omega_{2n}(s) = \frac{1}{n!}(-\frac{1}{2}a^*(K))^n\Omega_0(s)$. For

$n = 1$ we have already proved the formula. Let $n \in \mathbf{N}$ and suppose that $\Omega_{2n}(s) = \frac{1}{n!}(-\frac{1}{2}a^*(K))^n\Omega_0(s)$ then

$$\begin{aligned} a^*(K)a(g)a^*(K)^n\Omega &= 2n\, a^*(K)a^*(Kg)a^*(K)^{n-1}\Omega \\ &= 2n\, a^*(Kg)a^*(K)^n\Omega \end{aligned}$$

where the first equality is a result of the spectral resolution of K and the second equality is a consequence of $[a^*(Kg), a^*(K)] = 0$. So $n \cdot a^*(Kg)\Omega_{2n}(s)$ $= \frac{1}{2}a^*(K)a(g)\Omega_{2n}(s) = (\frac{1}{2}a(g)a^*(K) - a^*(Kg))\,\Omega_{2n}(s)$, by formula (1.8), and then

$$a^*(Kg)\Omega_{2n}(s) = \frac{1}{n+1}\cdot\frac{1}{2}a(g)a^*(K)\Omega_{2n}(s)$$

Hence

$$\begin{aligned} 0 &= a(g)\Omega_{2(n+1)}(s) + a^*(Kg)\Omega_{2n}(s) \\ &= a(g)(\Omega_{2(n+1)}(s) + \frac{1}{n+1}\cdot\frac{1}{2}a^*(K)\Omega_{2n}(s)) \end{aligned}$$

for all $g \in \mathcal{H}$, from which it follows that

$$\begin{aligned} \Omega_{2(n+1)}(s) &= \frac{1}{n+1}(-\frac{1}{2}a^*(K))\,\Omega_{2n}(s) \\ &= \frac{1}{(n+1)!}(-\frac{1}{2}a^*(K))^{n+1}\Omega_0(s) \end{aligned}$$

proving the desired formula for $n+1$. Thus

$$\begin{aligned} \Omega_s &= \oplus_{n=0}^{\infty}\frac{1}{n!}(-\frac{1}{2}a^*(K))^n\Omega_0(s) \\ &= e^{-\frac{1}{2}a^*(K)}\Omega_0(s) \\ &= c(s)e^{-\frac{1}{2}a^*(K)}\Omega \end{aligned}$$

since $\Omega_0(s) = c(s)\cdot\Omega$. This formula allow us to get a differential equation for $c(s)$, as follows

$$\begin{aligned} c'(s) &= \frac{d}{ds}\langle\Omega, \Omega_s\rangle = \langle\Omega, dU(A)\Omega_s\rangle \\ &= -\langle dU(A)\Omega, \Omega_s\rangle = \frac{1}{2}\langle a^*(A_2)\Omega, \Omega_s\rangle \\ &= \frac{1}{2}c(s)\langle\mathcal{A}_2, e^{-\frac{1}{2}a^*(K)}\Omega\rangle = -\frac{1}{4}c(s)\langle\mathcal{A}_2, \mathcal{K}\rangle \\ &= \frac{1}{2}\mathrm{Tr}(KA_2)\cdot c(s) \end{aligned}$$

where we have used formula (1.11), the just derived formula for Ω_s above and that $dU(A)\Omega = -\frac{1}{2}a^*(A_2)\Omega$. So $c(s)$ is given by the above differential equation and the fact that $c(0) = \|\Omega\|^2 = 1$. Notice that $K = T_2T_1^{-1} = (e^{sA})_2(e^{sA})_1^{-1}$ depends on $s \in \mathbf{R}$. Put $V_s = e^{-sA_1}(e^{sA})_1$, then is

$$\begin{aligned}\frac{d}{ds}V_s &= e^{-sA_1}(-A_1)(e^{sA})_1 + e^{-sA_1}(Ae^{sA})_1 \\ &= e^{-sA_1}(-A_1)(e^{sA})_1 + e^{-sA_1}(A_1(e^{sA})_1 + A_2(e^{sA})_2) \\ &= e^{-sA_1}A_2(e^{sA})_2\end{aligned}$$

and for s so small that V_s is close to the identity I and therefore invertible, we get

$$\begin{aligned}V_s^{-1}\frac{d}{ds}V_s &= (e^{sA})_1^{-1}e^{sA_1}e^{-sA_1}A_2(e^{sA})_2 \\ &= (e^{sA})_1^{-1}A_2(e^{sA})_2 = T_1^{-1}A_2T_2\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{ds}\mathrm{Tr}(\log V_s) &= \mathrm{Tr}(V_s^{-1}\frac{d}{ds}V_s) = \mathrm{Tr}(T_1^{-1}A_2T_2) \\ &= \mathrm{Tr}(A_2T_2T_1^{-1}) = \mathrm{Tr}(A_2K) = \overline{\mathrm{Tr}(KA_2)}\end{aligned}$$

where we have used that V_s^{-1} and $\frac{d}{ds}V_s$ commute (only) under the trace symbol, since both are complex linear operators, as is the case for T_1^{-1} and A_2T_2, and that $\overline{\mathrm{Tr}(KA_2)} = \mathrm{Tr}(A_2K)$, due to $\overline{\langle e_j, KA_2e_j\rangle} = \langle e_j, A_2Ke_j\rangle$, for an arbitrary basis vector $e_j \in \mathcal{H}$. Then we may write the differential equation as

$$c'(s) = \frac{1}{2}\frac{d}{ds}\overline{\mathrm{Tr}(\log(V_s))} \cdot c(s)$$

which has the solution

$$c(s) = \kappa \cdot e^{\frac{1}{2}\overline{\mathrm{Tr}(\log(V_s))}} = \kappa \cdot (\overline{\det(V_s)})^{\frac{1}{2}} = \kappa \cdot (\det(V_{-s}))^{\frac{1}{2}}$$

since $c(-s) = \overline{c(s)}$, and we have used that the determinant of V_s exists, since $V_s - I = \int_0^s e^{-tA_1}A_2(e^{tA})_2\,dt$ is a trace-class operator, because A_2 and $(e^{tA})_2$ are both Hilbert-Schmidt operators (see for example [R-S 3, p. 322]] or [Ar, p. 124]). Finally it follows that $\kappa = 1$, since $c(0) = 1$, so

$$c(s) = (\det(V_{-s}))^{\frac{1}{2}}$$

giving an explicit formula for $c(s)$, as claimed. We summarize the above as follows.

Theorem 8 *The vacuum functional* $c(s) = \langle \Omega, U(e^{sA})\Omega \rangle$ *for* $A \in \mathfrak{o}_2(\mathcal{H})$ *and* s *in a neighbourhood of zero, where* $U(\cdot)$ *denote the spin representation, is simply*

$$c(s) = (\det(V_{-s}))^{\frac{1}{2}}$$

where $V_{-s} = e^{sA_1}(e^{-sA})_1 = I - \int_0^s e^{tA_1} A_2 (e^{-tA})_2 \, dt$.

Proof. A imediately consequence of the above. ∎

Since the (restricted) unitary group can be realized as a subgroup of the (restriced) orthogonal group, we may study the restriction of the spin representation to the restricted unitary group. We do this in the next section.

1.5 The spin representation of the restricted unitary group.

As mentioned at the end of the former section, we will now consider the restriction of the spin representation to the restricted unitary group, partly because we get a nice explicit expression for the Lie algebra cocycle, which we will use later on in an application of the theory on a loop group. This is allowed by the fact that we may realize the restricted unitary group as a subgroup of the restricted orthogonal group.

The results of this section can be found in [Lu 2, section 6]. We bring a detailed elaboration of this below. Some of the ideas can be found in [P-S, chapter 6] and [Mi, chapter 6] too.

Let P be an orthogonal projection on the Hilbert space $\mathcal{H}$ and let $\mathcal{U}(\mathcal{H})$ denote the unitary group on $\mathcal{H}$. We define the restricted unitary group $\mathcal{U}_2(\mathcal{H},P)$ on $\mathcal{H}$ by

$$\mathcal{U}_2(\mathcal{H},P) = \{V \in \mathcal{U}(\mathcal{H}) : [P,V] \in \mathbf{L}_2(\mathcal{H})\}$$

where $\mathbf{L}_2(\mathcal{H})$, as earlier, denote the Hilbert-Schmidt operators on $\mathcal{H}$. We wish to say a few things about the corresponding "pre-Lie-algebra", which we denote $u_2(\mathcal{H},P)$. We demand that the elements of $u_2(\mathcal{H},P)$, through the exponential mapping, define unitary one-parameter groups in $\mathcal{U}_2(\mathcal{H},P)$, in resemblance with the preceding sections. Suppose $A \in u_2(\mathcal{H},P)$, and consider $V_s = e^{sA}$. From the unitarity of V_s, it follows, by taking the s-derivative at $s=0$, that A is skew-selfadjoint, $A^* = -A$ (and that A is complex linear). Of course, it would be convenient if $[P,V_s] \in \mathbf{L}_2(\mathcal{H})$, where $V_s = e^{sA}$, implies that $[P,A] \in \mathbf{L}_2(\mathcal{H})$, this is, however, not the case, since $\mathbf{L}_2(\mathcal{H})$ is closed only with respect to the Hilbert-Schmidt topology, and not the uniform topology. We define

$$u_2(\mathcal{H},P) = \{A \in \mathbf{L}(\mathcal{H}) : A^* = -A, [P,A] \in \mathbf{L}_2(\mathcal{H})\}$$

where $\mathbf{L}(\mathcal{H})$ denotes the bounded linear operators on $\mathcal{H}$, later on we may want to enlarge our choice of "pre-Lie-algebra" to unbounded operators.

It is evident that e^{sA} defines a unitary operator on $\mathcal{H}$, for $A \in u_2(\mathcal{H},P)$, so the non-trivial part, in proving that $e^{sA} \in \mathcal{U}_2(\mathcal{H},P)$, is to prove that

$[P, e^{sA}]$ is Hilbert-Schmidt, for any $A \in u_2(\mathcal{H}, P)$. Split A into two parts

$$B = (I-P)A(I-P) + PAP$$

and

$$C = PA(I-P) + (I-P)AP$$

such that $A = B + C$. It follows that both B and C is skew-selfadjoint and that C is Hilbert-Schmidt, since $PA(I-P) = P[P, A]$ and $[P, A]$ is Hilbert-Schmidt by assumption. Moreover $[P, B] = 0$. Define $V(s) = e^{-sB}e^{sA}$, $s \in \mathbf{R}$, which is a unitary operator. We now show that $V(s) - I$ may be written as a series, which converge in the Hilbert-Schmidt topology, uniformly in s, on compact sets in $\mathbf{R}$. Observe that

$$\begin{aligned}
\frac{d}{ds}V(s) &= e^{-sB}(A-B)e^{sA} \\
&= e^{-sB}Ce^{sA} \\
&= C(s)V(s)
\end{aligned}$$

where $C(s) = e^{-sB}Ce^{sB}$. Integration then gives

$$V(s) = I + \int_0^s C(t)V(t)\,dt$$

since $V(0) = I$. Put $V_0(s) = I$ and iterate the equation by putting

$$V_{n+1}(s) = \int_0^s C(t)V_n(t)\,dt$$

for $n \in \mathbf{N} \cup \{0\}$. Then

$$\|V_1(s)\|_{HS} \leq \|C\|_{HS} \cdot |\int_0^s dt| = \|C\|_{HS} \cdot |s|$$

and

$$\begin{aligned}
\|V_{n+1}(s)\|_{HS} &= \|\int_0^s C(t)V_n(t)\,dt\|_{HS} \\
&\leq |\int_0^s \|C(t)V_n(t)\|_{HS}dt| \\
&\leq |\int_0^s \|C(t)\|_{HS}\|V_n(t)\|_{HS}dt| \\
&= \|C\|_{HS} \cdot |\int_0^s \|V_n(t)\|_{HS}dt|
\end{aligned}$$

for $n \in \mathbf{N}$, since $\|C(t)\|_{HS} = \|C\|_{HS}$. Assuming that $\|V_n(s)\|_{HS} \leq \|C\|_{HS}^n \frac{|s|^n}{n!}$, then

$$\begin{aligned} \|V_{n+1}(s)\|_{HS} &\leq \|C\|_{HS} \cdot \left| \int_0^s \|V_n(t)\|_{HS} dt \right| \\ &\leq \|C\|_{HS}^{n+1} \cdot \frac{1}{n!} \cdot \left| \int_0^s |t|^n dt \right| \\ &= \|C\|_{HS}^{n+1} \cdot \frac{|s|^{n+1}}{(n+1)!} \end{aligned}$$

Thus

$$\| \sum_{n=1}^{\infty} V_n(s) \|_{HS} \leq \sum_{n=1}^{\infty} \|V_n(s)\|_{HS} \leq e^{\|C\|_{HS} \cdot |s|}$$

So the series $\sum_{n=1}^{\infty} V_n(s)$ converges in the Hilbert-Schmidt topology, uniformly in s on compact sets in $\mathbf{R}$. Moreover $\sum_{n=0}^{\infty} V_n(s)$ is a solution to the integral equation, determining $V(s)$. Hence $V(s) - I = \sum_1^{\infty} V_n(s)$ is Hilbert-Schmidt.

Since $[P, B] = 0$, it then follows that $Pe^{sA}(I - P) = Pe^{sB}V(s)(I - P)$ $= e^{sB}P(V(s) - I)(I - P)$ is Hilbert-Schmidt, because

$$\begin{aligned} \|Pe^{sA}(I - P)\|_{HS} &= \|e^{sB}P(V(s) - I)(I - P)\|_{HS} \\ &= \|P(V(s) - I)(I - P)\|_{HS} \\ &\leq \|V(s) - I\|_{HS} \\ &< \infty \end{aligned}$$

for s in a compact subset of $\mathbf{R}$. Hence $[P, e^{sA}] = Pe^{sA}(I - P) - (I - P)e^{sA}P$ is Hilbert-Schmidt, so $e^{sA} \in \mathcal{U}_2(\mathcal{H}, P)$.

Now, let Γ be an involution on $\mathcal{H}$ commuting with P (as in section 1.2 such anti-unitary involutions do exist). Put $I_P = I - P + \Gamma P$ and observe that $I_P{}^2 = (I - P)^2 + (\Gamma P)^2 = I - P + \Gamma^2 P = I$, so the real linear operator I_P is invertible on $\mathcal{H}$. Moereover

$$\begin{aligned} \tau(I_P f, I_P g) &= \tau((I - P)f, g) + \tau(\Gamma P f, \Gamma P g) \\ &= \tau((I - P)f, g) + \tau(Pg, Pf) \\ &= \tau(f, g) \end{aligned}$$

for all $f, g \in \mathcal{H}$, where $\tau(\cdot,\cdot)$, as earlier, denotes $Re\langle\cdot,\cdot\rangle$. This means that $I_P \in \mathcal{O}(\mathcal{H})$. By use of this, the restricted unitary group $\mathcal{U}_2(\mathcal{H}, P)$ can be realized as a subgroup of the restricted orthogonal group $\mathcal{O}_2(\mathcal{H})$. For $V \in \mathcal{U}_2(\mathcal{H}, P)$ we put $V_P = I_P V I_P$, then $V \to V_P$ defines a representation of $\mathcal{U}_2(\mathcal{H}, P)$ in $\mathcal{O}_2(\mathcal{H})$, since V_P evidently is real linear and invertible, in fact $V_P^{-1} = I_P V^* I_P$, and a direct calculation shows that $\tau(V_P f, V_P g) = \tau(f, g)$, using the unitarity of V and that $I_P \in \mathcal{O}(\mathcal{H})$, so $V_P \in \mathcal{O}(\mathcal{H})$. Finally

$$\begin{aligned} V_P &= I_P V I_P \\ &= (I-P)V(I-P) + P\Gamma V \Gamma P + (I-P)V\Gamma P + \Gamma P V(I-P) \\ &= (I-P)V(I-P) + P\Gamma V\Gamma P + \Gamma P[P,V] - [P,V]\Gamma P \\ &= (V_P)_1 + (V_P)_2 \end{aligned}$$

where $(V_P)_1 = (I-P)V(I-P) + P\Gamma V\Gamma P$ is the linear part of V and the anti-linear part $(V_P)_2 = \Gamma P[P,V] - [P,V]\Gamma P = [\Gamma P, [P,V]]$ is Hilbert-Schmidt, due to the fact that $[P,V] \in \mathbf{L}_2(\mathcal{H})$, since $V \in \mathcal{U}_2(\mathcal{H}, P)$. Hence $V_P \in \mathcal{O}_2(\mathcal{H})$.

This allows us to construct the spin representation of the restricted unitary group, and the corresponding subgroup of $Spin_2(\mathcal{H})$ will be denoted $Spin_2(\mathcal{H}, P)$.

Define $U_P(V) = U(V_P)$, for any $V \in \mathcal{U}_2(\mathcal{H}, P)$, where $U(\cdot)$ is the spin representation, defined in section 1.4, $U(V_P)$ is well-defined, since $V_P \in \mathcal{O}_2(\mathcal{H})$. Moreover, we put $a_P(f) = a((I-P)f) + a^*(\Gamma P f)$, for $f \in \mathcal{H}$, which clearly is anti-linear, and $f \to a_P(f)$, evidently gives a representation of the *CAR*-algebra labelled by P, since $(I-P)$ and ΓP are Bogoliubov transformations (see section 1.2, just above theorem 1). Then

$$\pi_{I_P}(f) = \pi(I_P f) = \frac{1}{\sqrt{2}}(a_P(f) + a_P{}^*(f))$$

for $f \in \mathcal{H}$, gives a representation of the Clifford algebra, where we have used that $I_P{}^{-1} = I_P$. By use of formula (1.12) and the fact that $I_P V = V_P I_P$ it follows that

$$\begin{aligned} \pi_{I_P}(Vf) &= \pi(I_P V f) = \pi(V_P I_P f) \\ &= U(V_P)\pi(I_P f)U(V_P)^{-1} \\ &= U_P(V)\pi_{I_P}(f)U_P(V)^{-1} \end{aligned}$$

and then is

$$a_P(Vf) = U_P(V)a_P(f)U_P(V)^{-1}$$

for $f \in \mathcal{H}$ and $V \in \mathcal{U}_2(\mathcal{H}, P)$, of course we also have the analogous formula for $a_P{}^*(\cdot)$.

Let $A \in u_2(\mathcal{H}, P)$ and define $dU_P(A)$ as the generator of the unitary one-parameter group $U_P(e^{sA}) = U(I_P e^{sA} I_P)$. In the case where A is bounded, as in the above choice of pre-Lie-algebra $u_2(\mathcal{H}, P)$, it follows that $I_P e^{sA} I_P = e^{sA_P}$ and then is $dU_P(A) = dU(A_P)$. For $A = iI$, which is in $u_2(\mathcal{H}, P)$, is $U_P(e^{siI}) = e^{sdU_P(iI)} = e^{sdU((iI)_P)}$ and $(iI)_P = i(I - 2P)$, which, in fact, is complex linear, so

$$dU_P(iI) = dU((i(I - 2P))_1) = idU(I - 2P)$$

where the last equality follows because the generator reduce to the linear Fock-Cook generator. Then we may define the socalled charge operator Q as the selfadjoint operator $Q = -idU_P(iI) = dU(I - 2P)$, which is well-defined at least on $\mathcal{D}$. To discuss the spectrum of Q we define $\mathcal{H}_- = P\mathcal{H}$ and $\mathcal{H}_+ = (I - P)\mathcal{H} = \mathcal{H}_-^\perp$ such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and let $\{e_j : j \in M_\pm\}$ denote an arbitrary choice of orthonormal basis for respectively $\mathcal{H}_+$ and $\mathcal{H}_-$. Notice that this splitting of $\mathcal{H}$ into $\mathcal{H}_+ \oplus \mathcal{H}_-$ equips $\mathcal{H}$ with a polarization, defined by the unitary operator $J = I - 2P$, on $\mathcal{H}$, which is +1 on $\mathcal{H}_+$ and -1 on $\mathcal{H}_-$, since $JP = -P$ and $J(I - P) = (I - P)$. Hence the demanded $[P, V] \in \mathbf{L}_2(\mathcal{H})$ is equivalent to $[J, V] \in \mathbf{L}_2(\mathcal{H})$, which is the formulation used in [P-S, p. 80]. In a more physical language, J is a one-particle charge operator corresponding to the case where $\mathcal{H}_\pm$ is the space representing particle with respectively positive (+) and negative (-) charge, so $Je_k = q_k \cdot e_k$, where $q_k = \pm 1$ for $k \in M_\pm$, respectively, i.e. q_k is the sign of the charge of the particle described by the state vector e_k. Then $Q = dU(J)$ on $\mathcal{D}$. Now, it follows that $\Omega \in \mathcal{D}$ and $Q\Omega = 0$, by definition. Moreover $e_{j_1} \wedge \cdots \wedge e_{j_n} \in \mathcal{D}$ and

$$\begin{aligned} Q(e_{j_1} \wedge \cdots \wedge e_{j_n}) &= \sum_{k=1}^{n} e_{j_1} \wedge \cdots \wedge Je_{j_k} \wedge \cdots \wedge e_{j_n} \\ &= (\sum_{k=1}^{n} q_{j_k}) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_n}) \end{aligned}$$

notice that $\sum_{k=1}^{n} q_{j_k} \in \mathbf{Z}$. Evidently, we have

$$\|Q(e_{j_1} \wedge \cdots \wedge e_{j_n})\| \leq n \cdot \|e_{j_1} \wedge \cdots \wedge e_{j_n}\|$$

so $\mathcal{D}(N) \subset \mathcal{D}(Q)$, where N denote the number operator, discussed in the end of section 1.3. Observe that the eigenvalue corresponding to $e_{j_1} \wedge \cdots \wedge e_{j_n}$

belongs to $\{-n, -n+2, \ldots, n-2, n\}$, where $n \in \mathbf{N}$, and all possibilities can occur, of course dependent of the choice of the eigenvector. This means that the set of eigenvalues of Q at least is $\mathbf{Z}$ and that the eigenvectors corresponding to a fixed eigenvalues $q \in \mathbf{Z}$ are all in the subspace spanned by those product basis vectors, made up of anti-symmetric tensor product of one-particle vectors with q particle more from $\mathcal{H}_{sign(q)}$ than from the orthogonal complement $\mathcal{H}^{\perp}_{sign(q)} = \mathcal{H}_{-sign(q)}$ (here the sign of $q = 0$ is optional). As a consequence, there are infinitely many eigenvectors to each eigenvalue, $q \in \mathbf{Z}$. Moreover, Q is evidently unbounded, since its spectrum is unbounded. Up to now we have proved that the spectrum of Q include $\mathbf{Z}$, we shall now prove that the spectrum of Q indeed is $\sigma(Q) = \mathbf{Z}$. For all $s \in \mathbf{R}$ is $e^{siQ} = U_P(e^{siI})$ $= U_P(e^{siI(1+2\pi)}) = e^{siQ} \cdot e^{i2\pi Q}$, then $e^{i2\pi Q} = I$, which implies that $\sigma(Q) \subset \mathbf{Z}$, hence $\sigma(Q) = \mathbf{Z}$. Moreover, for $\lambda \notin \mathbf{Z}$, is the range of $Q - \lambda$ dense in $\mathcal{F}_\wedge(\mathcal{H})$ and

$$\|(Q - \lambda \cdot I)F\| \geq c_\lambda \cdot \|F\|$$

where $c_\lambda = dist(\lambda, \mathbf{Z}) > 0$, so $(Q-\lambda)^{-1}$ is well-defined with bound c_λ^{-1}. Notice that the spectrum consists of eigenvalues only. For each $q \in \mathbf{Z} = \sigma(Q)$, let $\mathcal{H}_q \subset \mathcal{F}_\wedge(\mathcal{H})$ denote the eigenspace of Q corresponding to the eigenvalue q. Then is $\mathcal{H}_q \perp \mathcal{H}_{q'}$, for $q \neq q'$, both in $\mathbf{Z}$. Hence we are able to decompose $\mathcal{F}_\wedge(\mathcal{H})$ in the following way

$$\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{q \in \mathbf{Z}} \mathcal{H}_q$$

called the charge gradation of $\mathcal{F}_\wedge(\mathcal{H})$. If $A \in u_2(\mathcal{H}, P)$, then Q commute with $dU_P(A)$, by use of (1.13), (1.14) and the fact that $[A_P, B_P] = I_P[A, B]I_P = [A, B]_P$, for $A, B \in u_2(\mathcal{H}, P)$, since

$$\begin{aligned}
[dU_P(A), Q] &= [dU(A_P), dU(J)] \\
&= dU([A_P, J]) + \omega(A_P, J) \cdot I \\
&= dU([A, iI]_P) - i\omega(A_P, (iI)_P) \cdot I \\
&= 0 - \frac{i}{2}\mathrm{Tr}([(A_P)_2, (I_P iI_P)_2]) \cdot I \\
&= 0
\end{aligned}$$

because $I_P iI_P = i(I - 2P) = iJ$ is linear. Notice that this argument only holds for A bounded. However, Q does not commute with all operators $U_P(V) = U(V_P)$, $V \in \mathcal{U}_2(\mathcal{H}, P)$ (as can be seen from the spin representation of the charge group C, which will be treated later on). The above charge gradation and the fact that Q commute with $dU_P(A)$, and then with $U_P(e^{sA})$, for $A \in u_2(\mathcal{H}, P)$, implies that the operator $U_P(e^{sA})$ maps $\mathcal{H}_q$ into $\mathcal{H}_q$. That the operator $U_P(e^{sA})$, for $A \in u_2(\mathcal{H}, P)$, leaves $\mathcal{H}_q$ invariant, then means that $U_P(e^{sA})$ conserves the charge. But, since not all operators $U_P(V)$, for $V \in \mathcal{U}_2(\mathcal{H}, P)$, commutes with Q, they do not leave $\mathcal{H}_q$ invariant, and therefore they do not, in general, conserve the charge.

Let us calculate an explicit formula for the Lie algebra cocycle, in the case where $A, B \in u_2(\mathcal{H}, P)$, by use formula (1.11) in theorem 4, as follows

$$\begin{aligned}
&\langle \Omega, dU_P(A) dU_P(B)\, \Omega\rangle \\
&= \langle \Omega, dU(A_P) dU(B_P)\, \Omega\rangle \\
&= \frac{1}{2}\mathrm{Tr}((B_P)_2 (A_P)_2) \\
&= \frac{1}{2}\mathrm{Tr}((I-P)BPA(I-P) + \Gamma PB(I-P)AP\Gamma) \\
&= \frac{1}{2}\mathrm{Tr}((I-P)BPA(I-P)) + \frac{1}{2}\mathrm{Tr}((PB(I-P)AP)^*) \\
&= \frac{1}{2}\mathrm{Tr}(PA(I-P)BP) + \frac{1}{2}\mathrm{Tr}(PA(I-P)BP) \\
&= \mathrm{Tr}(PA(I-P)BP)
\end{aligned} \tag{1.15}$$

where we have used that

$$\begin{aligned}
(C_P)_2 &= (I_P C I_P)_2 \\
&= ((I-P)C(I-P) + \Gamma PC\Gamma P + (I-P)C\Gamma P + \Gamma PC(I-P))_2 \\
&= (I-P)C\Gamma P + \Gamma PC(I-P)
\end{aligned}$$

for $C = A, B$, that $\Gamma^2 = I$, that $\mathrm{Tr}(\Gamma C \Gamma) = \mathrm{Tr}(C^*)$, that $A^* = -A$, $B^* = -B$, that $(I-P)BP = [B, P]P$ and that $PA(I-P) = P[P, A]$ both are linear Hilbert-Schmidt operators such that

$$\mathrm{Tr}((I-P)BPA(I-P)) = \mathrm{Tr}(PA(I-P)BP)$$

Hence, from formula (1.14), in theorem 6, we get the following explicit expression for the Lie algebra cocycle

$$\begin{aligned}
\omega(A_P, B_P) &= \langle \Omega, [dU(A_P), dU(B_P)]\,\Omega\rangle \\
&= -\frac{1}{2}\mathrm{Tr}([(A_P)_2, (B_P)_2]) \\
&= \mathrm{Tr}(PA(I-P)BP) - \mathrm{Tr}(PB(I-P)AP)
\end{aligned} \tag{1.16}$$

so $\omega_P(A, B) = \mathrm{Tr}(PA(I-P)BP) - \mathrm{Tr}(PB(I-P)AP)$, where $\omega_P(A, B) = \omega(A_P, B_P)$ and the analogue of formula (1.13) then reads

$$[dU_P(A), dU_P(B)] = dU_P([A, B]) + \omega_P(A, B) \cdot I \tag{1.17}$$

for $A, B \in u_2(\mathcal{H}, P)$.

We end our treatment of the spin representation, based on the CAR-algebra and its Fock representation, by the following diagram, reflecting our construction.

$$\begin{tikzcd}
\mathcal{U}_2(\mathcal{H}, P) \arrow[rrr, "i_p"] \arrow[dddr, "U_p"'] & & & \mathcal{O}_2(\mathcal{H}) \arrow[dddl, "U"] \\
& u_2(\mathcal{H}, P) \arrow[ul, "exp"'] \arrow[r, "i_p : A \to A_p"] \arrow[d, "dU_p = dU \circ i_p"] & o_2(\mathcal{H}) \arrow[ur, "exp"'] \arrow[d, "dU"] & \\
& spin_2(\mathcal{H}, P) \arrow[r, phantom, "\subset"] \arrow[d, "exp"] & spin_2(\mathcal{H}) \arrow[d, "exp"'] & \\
& Spin_2(\mathcal{H}, P) \arrow[r, phantom, "\subset"] & Spin_2(\mathcal{H}) &
\end{tikzcd}$$

We return to some applications in a later chapter. Before doing so, we will make an analogous discussion of the CCR-algebra. However we conclude this chapter with an appendix-like section on a discussion of different formalisms.

1.6 A note on the formalism.

We have several times referred to other authors, using another formalism, see for example [Ar]. This appendix-like section is devoted to the transformation between our formalism, which is consistent with the formalism used in some reference papers, such as [Lu 2], [B-R 2] and [V-Z], and the formalism used by some other, such as [Ar] and [Ya]. To clarify the discussion, we will only consider the transformation between the formalism used in this thesis and the formalism used by Araki, in [Ar]. The discussion of this section doesn't seem to appear anywhere else.

Let $\mathcal{H}$ denote a complex Hilbert space and let $\mathcal{H}^*$ denote the conjugate Hilbert space, that is $\mathcal{H}^*$ denote the same set as $\mathcal{H}$, with the same addition rule, but with the complex conjugated scalar multiplication, $\mathbf{C} \times \mathcal{H} \to \mathcal{H}$ (relative to $\mathcal{H}$) given by $\lambda \cdot f = \overline{\lambda} f$ and the complex conjugated inner product, $\mathcal{H} \times \mathcal{H} \to \mathbf{C}$ given by $\langle f, g\rangle_{\mathcal{H}^*} = \overline{\langle f, g\rangle_{\mathcal{H}}} = \langle g, f\rangle_{\mathcal{H}}$. Put $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^*$ and write an $h \in \mathcal{K}$ as $h = h_1 + h_2$, where $h_1 \in \mathcal{H}$ and $h_2 \in \mathcal{H}^*$. Then $\langle h, h'\rangle_{\mathcal{K}} = \langle h_1, h_1'\rangle_{\mathcal{H}} + \overline{\langle h_2, h_2'\rangle_{\mathcal{H}}}$. This is the Hilbert space used in [Ar].

Define a orthogonal projection P onto $\mathcal{H}$ on $\mathcal{K}$ by $Ph = h_1$, for any $h \in \mathcal{K}$, where $h = h_1 + h_2$ refers to the above splitting. We also define a complex conjugation on $\mathcal{K}$ by $\Gamma(h_1, h_2) = (h_2, h_1)$, for $h_1 \in \mathcal{H}$, $h_2 \in \mathcal{H}^*$, where we write (h_1, h_2) for $h = h_1 + h_2$. Notice that $\Gamma\lambda h = \overline{\lambda}\Gamma h$. We may write $\Gamma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Observe that $P + \Gamma P \Gamma = I$, hence P is what Araki call a basis projection. Moreover Araki define $Re_\Gamma\mathcal{K} = \{h = (h_1, h_2) \in \mathcal{K} : h_1 = h_2\}$ $= \{h \in \mathcal{K} : \Gamma h = h\}$ and then is $\langle (h,h), (h',h')\rangle_{\mathcal{K}} = 2 \cdot Re\langle h, h'\rangle_{\mathcal{H}}$, for $(h,h), (h',h') \in Re_\Gamma\mathcal{K}$. Then Araki define a generalized "Clifford" operator (relative to our) by

$$\begin{aligned} \pi_P(B(h)) &= a(Ph)^* + a(P\Gamma h) \\ &= a((h_1, 0))^* + a((h_2, 0)) \\ &= a(h_1)^* + a(h_2) \end{aligned}$$

in the Fock representation (here π_P denote the Fock representation given by P). However, on $Re_\Gamma\mathcal{K}$, the generalized "Clifford" operator match with our "Clifford" operator $\pi(h) = a(h)^* + a(h)$, where π denote our "Clifford" operator, defined only on the space $Re_\Gamma\mathcal{K}$, which corresponds to our Hilbert space (which we in preceding sections denoted $\mathcal{H}$). The advantage in Araki's formalism, is that our orthogonal transformations corresponds to unitary operators, which commute with Γ, and then one are able to use spectral theory directly. However, this formalism pays the price, to deal with a Hilbert space, twice as big as our physical Hilbert space, $Re_\Gamma\mathcal{K}$.

A (unitary) operator U on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^*$ may be written as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The requirement that $[U, \Gamma] = 0$, then gives that

$$B = C = W = PU(I - P) + (I - P)UP$$

and that

$$A = D = V = PUP + (I - P)U(I - P)$$

The unitarity of U then gives

$$V^*V + W^*W = I = VV^* + WW^*$$

and

$$V^*W + W^*V = 0 = WV^* + VW^*$$

which is equivalent to our decomposition of a orthogonal transformation $T = T_1 + T_2$, with $T_1 = V$ and $T_2 = W$, observe that $T = V + W$ in fact becomes an orthogonal transformation on $Re_\Gamma\mathcal{K}$ and that $W : \mathcal{H} \oplus \mathcal{H}^* \to (\mathcal{H} \oplus \mathcal{H}^*)^*$ is complex linear, corresponding to a complex anti-linear operator on $Re_\Gamma\mathcal{K}$. Hence

$$U = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix}$$

explains the connection of the transformation between operators used in the respective formalisms.

Chapter 2

The Metaplectic Representation of the Infinite Dimensional Symplectic Group based on the CCR Algebra in its Fock Representation

2.1 The canonical commutation relations and the CCR-algebra.

In this section we will introduce the canonical commutation relations and the CCR-algebra. Although the CCR-algebra has many properties analogous to the CAR-algebra treated in chapter 1, there is a lack of norm continuity in the definition of the CCR-algebra. This lack of norm continuity is related to the fact that creation and annihilation operators in any representation are unbounded. Physically this reflects the main difference between fermions, related to the CAR-algebra, obeying the Pauli principle (see chapter 1) and bosons, related to the CCR-algebra, which doesn't obey the Pauli principle. So in case of bosons there is no bound on the number of particles which can occupy a given physical state. This is reflected by the unboundedness of the annihilation and creation operators mentioned above. Usually one treats the boson case from a slightly different viewpoint, than the fermion case due to the previously mentioned qualitative difference between the two cases. One use the socalled Weyl operators which roughly speaking are the unitary operators constructed from the selfadjoint closure of the sum of the annihilation and creation operators to study a bosonic many-particle system. The Weyl operators fulfil the socalled Weyl form of the canonical commutation relations. The canonical commutation relations were introduced by Dirac (see [Dir]) in 1927 in the context of radiation theory in physics. However, Cook (see [Co]) were properbly the first to give a rigorous mathematical approach to the subject. It shall be mentioned that the abstract structure of creation and annihilation operators has been studied in the 1950's. In particular did Segal (see [Se,I.E.]) emphasize the C^*-algebraic structure of the CCR-algebra and introduced severel new feature and results.

In the following sections we will use the concrete realization of the canonical commutation relations called the Fock representation of the canonical commutation relations. We will therefore focus on this representation. We include, however, a general discussion of the CCR-algebra serving as a natural frame similar to the case of the CAR-algebra.

This section is based on chapter 1 in this thesis, [Co], [Lu 2], [Ya] and [B-R 2]. We will use a notation similar to that of chapter 1, but common symbols does in general mean something else.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{A}$ an abstract $*$-algebra. We call $\mathcal{A}$ the $*$-algebra generated by $\{a(f) : f \in \mathcal{H}\}$, if there is an anti-linear mapping $a : \mathcal{H} \mapsto \mathcal{A}$ such that $\{a(f) : f \in \mathcal{H}\}$ together whith the unit I generates the $*$-algebra $\mathcal{A}$ and fulfil *the canonical commutation relations*

$$\begin{aligned} [a(f), a(g)] &= 0 \\ [a(f), a(g)^*] &= \langle f, g\rangle \cdot I \end{aligned} \tag{2.1}$$

for all $f, g \in \mathcal{H}$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$ and $[\cdot, \cdot]$ the usual commutator $[A, B] = AB - BA$. We will shortly refer to the canonical commutation relations given in (2.1) as the CCR.

Assume that we have a C^*-norm on $\mathcal{A}$, i.e. a norm fulfilling that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. The element $\pi(f) = a(f) + a(f)^* \in \mathcal{A}$ is selfadjoint with respect to the involution $*$. Elements of the form $\pi(f)$ then fulfil

$$\begin{aligned} [\pi(f), \pi(g)] &= [a(f), a(g)^*] + [a(f)^*, a(g)] \\ &= (\langle f, g\rangle - \langle g, f\rangle) \cdot I \\ &= 2 \cdot Im\, \langle f, g\rangle \cdot I \end{aligned}$$

Choose now $f, g \in \mathcal{H}$ such that $Im\, \langle f, g\rangle = \frac{c}{2} \neq 0$ then $[\pi(f), \pi(g)] = c \cdot I$. This is possible since for any $f \in \mathcal{H} \setminus \{0\}$, we may choose $g = if$ whereby $Im\, \langle f, g\rangle = \|f\|^2$ whence $c = 2 \cdot \|f\|^2 \neq 0$. For an arbitrary $k \in \mathbf{N}$ we therefore have that

$$[\pi(f), \pi(g)^k] = k \cdot c \cdot \pi(g)^{k-1}$$

by successive use of (2.1). So

$$\begin{aligned} k \cdot |c| \cdot \|\pi(g)\|^{k-1} &= k \cdot |c| \cdot \left\|\pi(g)^{k-1}\right\| \\ &= \left\| \left[\pi(f), \pi(g)^k\right] \right\| \\ &\leq 2 \cdot \|\pi(f)\| \cdot \left\|\pi(g)^k\right\| \\ &= 2 \cdot \|\pi(f)\| \cdot \|\pi(g)\|^k \end{aligned}$$

where the C^*-norm have been properly used repeatedly together with the selfadjointness of $\pi(g)$. Hence

$$\frac{1}{2} \cdot |c| \cdot k \leq \|\pi(f)\| \cdot \|\pi(g)\|$$

for all $k \in \mathbf{N}$, so at least one of $\pi(f)$ and $\pi(g)$ is unbounded contradicting the C^*-norm assumption. This means that we cannot turn $\mathcal{A}$ into a C^*-algebra.

It turns out (for details, see [Ya]) that there exists a normalized positive linear functional φ on $\mathcal{A}$, so we may apply the GNS construction to get a Hilbert space $\mathcal{F}$, a $*$-representation π of $\mathcal{A}$, i.e. a $*$-homomorphism from $\mathcal{A}$ into the set of linear (generally unbounded) operators on $\mathcal{F}$, and a generating vector Ω. Here $\mathcal{F}$ is the Fock space for the Fock representation π of $\mathcal{A}$ and Ω the vacuum vector. By construction $\varphi(A) = \langle \Omega, \pi(A)\Omega \rangle$, for each $A \in \mathcal{A}$. The GNS construction depends on φ. We will, however, not follow this track, but rather construct concrete realizations of the creation and annihilation operators on the symmetric Fock Hilbert space, and then show that it is in fact a representation of the $*$-algebra generated by $\{a(f) : f \in \mathcal{H}\}$. We will thus pursue the idea used in the construction of the Fock representation of the CAR-algebra (see section 1.2 of chapter 1).

The following discussion is an elaborated mixture of [B-R 2, section 5.2.1] and [Co, part I]. For each $f \in \mathcal{H}$ we define the linear operator $a^*(f)$ in the symmetric Fock Hilbert space $\mathcal{F}_\vee(\mathcal{H})$ as defined in section 1.1 of chapter 1, by

$$a^*(f)\Omega = f \tag{2.2}$$

and on product vectors by

$$a^*(f)\,(f_1 \vee \cdots \vee f_n) = f \vee f_1 \vee \cdots \vee f_n \tag{2.3}$$

We ought to write $a_0^*(f)$ instead of just $a^*(f)$ as it is a concrete realization ($*$-representation) of the abstract $*$-algebra $\mathcal{A}$ (which is shown below). We will, however, mainly consider this realization, so we discard the 0-indices, as we did in the case of the Fock representation of the CAR-algebra. Extension by linearity gives a densely defined linear operator $a^*(f)$ from $\vee^n\mathcal{H}$ to $\vee^{n+1}\mathcal{H}$, for each $n \in \mathbf{N} \cup \{0\}$. Analogously to the CAR-case, we call $a^*(f)$ the *creation operators*, for obvious reasons. The lack of a Pauli principle which is a consequence of the symmetry of product vectors in $\mathcal{F}_\vee(\mathcal{H})$, in contrast to the anti-symmetry of $\mathcal{F}_\wedge(\mathcal{H})$, allows us to have non-zero product vectors in $\mathcal{F}_\vee(\mathcal{H})$ of the form $F_n = f \vee \cdots \vee f \in \vee^n\mathcal{H}$, the n-fold (symmetric) tensor product of $f \in \mathcal{H}$ with itself. In contrast to the anti-symmetric case, the creation operators defined by (2.2) and (2.3) become unbounded, since

$$\begin{aligned}\|a^*(f)F_n\|^2 &= \|\overbrace{f\vee f\vee\cdots\vee f}^{(n+1)-times}\|^2 \\ &= (n+1)\cdot\|f\|^2\cdot\|\overbrace{f\vee f\vee\cdots\vee f}^{n-times}\|^2 \\ &= (n+1)\cdot\|f\|^2\cdot\|F_n\|^2\end{aligned}$$

where we have used the formula of the inner product on $\mathcal{F}_\vee(\mathcal{H})$ (see section 1.1) and that $F_n = \overbrace{f\vee\cdots\vee f}^{n-times} \in \vee^n(\mathcal{H})$. This evidently means that $a^*(f)$ is unbounded when extended by linearity to $\mathcal{F}_\vee(\mathcal{H})$. We will prove that in general

$$\|a^*(f)F_n\| \leq \sqrt{n+1}\cdot\|f\|\cdot\|F_n\| \tag{2.4}$$

for all $f\in\mathcal{H}$ and $F_n\in\vee^n\mathcal{H}$ where we already know that equality occurs for a certain choice of $F_n\in\vee^n\mathcal{H}$. For $f=0$ then $a^*(f)F_n=0$ and the claim follows trivially. For $f\neq 0$, we put $e_1=\frac{f}{\|f\|}$ and choose an orthonormal basis $\{e_j\}_{j=2}^\infty$ for $\{e_1\}^\perp$ in $\mathcal{H}$. Then the basis product vectors $E_{(i)_m}^{(k)_m} = e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}$, where $i_1<\cdots<i_m$ and $\sum_{j=1}^m k_j = n,\ k_j\in\mathbf{N}$, span a dense subspace in $\vee^n\mathcal{H}$ (see section 1.1). If we thus can prove the claim for all $F_n\in\vee^n\mathcal{H}$ of the form $\sum_{(i)_m,(k)_m} a_{(i)_m}^{(k)_m}E_{(i)_m}^{(k)_m}$ then the claim holds for all $F_n\in\vee^n\mathcal{H}$. A straight forward calculation using that $\left\|e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}\right\|^2 = \prod_{j=1}^m(k_j!)$ and that $\left\langle e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}, e_{j_1}^{l_1}\vee\cdots\vee e_{j_p}^{l_p}\right\rangle = 0$ if $e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}\neq e_{j_1}^{l_1}\vee\cdots\vee e_{j_p}^{l_p}$, where $k_1+\cdots+k_m = l_1+\cdots+l_p = n$ (see section 1.1 for further details) gives

$$\begin{aligned}&\left\|a^*(f)\sum a_{(i)_m}^{(k)_m}\cdot e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}\right\|^2 \\ &= \|f\|^2\cdot\left\|\sum a_{(i)_m}^{(k)_m}\cdot e_1\vee e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}\right\|^2 \\ &= \|f\|^2\cdot\sum\left|a_{(i)_m}^{(k)_m}\right|^2\cdot\begin{cases}1\cdot(k_1!)\cdot\ldots\cdot(k_m!) & \text{, for } 1<i_1<\cdots<i_m \\ (k_1+1)!\cdot(k_2!)\cdot\ldots\cdot(k_m!) & \text{, for } 1=i_1<\cdots<i_m\end{cases} \\ &\leq \|f\|^2\cdot\sum\left|a_{(i)_m}^{(k)_m}\right|^2\cdot(k_1+1)\cdot\prod_{j=1}^m(k_j!) \\ &= \|f\|^2\cdot(k_1+1)\cdot\left\|\sum a_{(i)_m}^{(k)_m}\cdot e_{i_1}^{k_1}\vee\cdots\vee e_{i_m}^{k_m}\right\|^2 \\ &\leq (n+1)\cdot\|f\|^2\cdot\|F_n\|^2\end{aligned}$$

proving our claim. Then $a^*(f)$, $f \in \mathcal{H}$ are unbounded operators in $\mathcal{F}_\vee(\mathcal{H})$ mapping $\vee^n\mathcal{H}$ into $\vee^{n+1}\mathcal{H}$. However, $a^*(f)$ is bounded as an operator from $\vee^n\mathcal{H}$ onto $\vee^{n+1}\mathcal{H}$. Observe that $a^*(f)$ is well-defined on $\mathcal{D}(N^{1/2}) = \{F = \oplus_{n=0}^\infty F_n : \sum_{n=0}^\infty n \cdot \|F_n\|^2 < \infty\}$, where N denotes the socalled number operator given by $N(\oplus_{n=0}^\infty F_n) = \oplus_{n=0}^\infty n \cdot F_n$. In fact $\mathcal{D}(N^{1/2})$ is the maximal domain for $a^*(f)$ on $\mathcal{F}_\vee(\mathcal{H})$. But in analogy with the anti-symmetric case, we may choose $\mathcal{D} = \oplus_{alg}(\vee^n\mathcal{H})$, the algebraic direct sum, as the dense domain of all $a^*(f)$, $f \in \mathcal{H}$. Indeed, $a^*(f)$ is well-defined on $\mathcal{D}$, since given any $F = \oplus_{n=0}^\infty F_n$ in $\mathcal{D}$ there exists an $N \in \mathbf{N}$ such that $F_n = 0$ for $n > N$, so $\|a^*(f)F\|^2 \leq \sum_{n=0}^N (n+1) \cdot \|f\|^2 \cdot \|F_n\|^2 \leq (N+1) \cdot \|f\|^2 \cdot \|F\|^2$. Moreover $\mathcal{D}$ is invariant under the action of $a^*(f)$, since $a^*(f) : \vee^n\mathcal{H} \mapsto \vee^{n+1}\mathcal{H}$ creates one particle, so if $F_n = 0$, for $n > N$ then $G_n = a^*(f)F_n = 0$, for $n > N+1$, where $G = a^*(f)F = \oplus_{n=0}^\infty G_n$. In fact, $\mathcal{D}$ is a dense set of analytic vectors for all $a^*(f)$, $f \in \mathcal{H}$. For arbitrary $F_n \in \vee^n\mathcal{H}$ and $f \in \mathcal{H}$ we have

$$\left\|(a^*(f))^k F_n\right\| \leq \sqrt{\frac{(n+k)!}{n!}} \cdot \|f\|^k \cdot \|F_n\|$$

by successive use of the above norm estimate and that $a^*(f)^m F_n \in \vee^{n+m}\mathcal{H}$. So for any $F = \oplus_{n=0}^\infty F_n \in \mathcal{D}$ there exists a $N \in \mathbf{N}$ such that $F_n = 0$ for $n > N$. Hence

$$\left\|(a^*(f))^k F\right\| \leq \sum_{n=0}^N \sqrt{\frac{(n+k)!}{n!}} \cdot \|f\|^k \cdot \|F_n\| \leq \sqrt{\frac{(N+k)!}{N!}} \cdot \|f\|^k \cdot \|F\|$$

Therefore

$$\sum_{k=0}^\infty \frac{\left\|(a^*(f))^k F\right\|}{k!} \cdot t^k \leq \|F\| \cdot \sum_{k=0}^\infty \frac{(t \cdot \|f\|)^k}{k!} \cdot \sqrt{\frac{(N+k)!}{N!}}$$

is finite for all $t \in \mathbf{R}_+$ by the ratio test. Since $\mathcal{D}$ is an invariant set of C^∞-vectors for $a^*(f)$, it follows that $\mathcal{D}$ is a set of analytic vectors for $a^*(f)$.

Now we are able to define annihilation operators $a(f)$, $f \in \mathcal{H}$, on $\mathcal{D}$ (one could choose to define $a(f)$ on $\mathcal{D}(N^{1/2})$). We define annihilation operators $a(f)$, $f \in \mathcal{H}$, as the adjoint operators of $a^*(f)$, i.e. $a(f) = a^*(f)^*$, for $f \in \mathcal{H}$. Since $a^*(f)$ is unbounded, so is $a(f)$, $f \in \mathcal{H}$, and we have to specify a domain of $a(f)$. We will prove that we may choose $\mathcal{D}$ as domain for all $a(f)$, $f \in \mathcal{H}$, and that $a(f)$ is given on product vectors in this domain by

$$a(f)\Omega = 0 \tag{2.5}$$

and

$$a(f)(f_1 \vee \cdots \vee f_n) = \sum_{i=1}^{n} \langle f, f_i \rangle \cdot f_1 \vee \cdots \vee f_{i-1} \vee f_{i+1} \vee \cdots \vee f_n \qquad (2.6)$$

Observe that $a(f)$ is bounded as an operator from $\vee^{n+1}\mathcal{H}$ to $\vee^n\mathcal{H}$ since $a^*(f)$ is, as an operator from $\vee^n\mathcal{H}$ to $\vee^{n+1}\mathcal{H}$. Since $\langle \Omega, a^*(f)F \rangle = 0$ for all $F \in \mathcal{D}$, it follows trivially that $\Omega \in \mathcal{D}(a(f))$ and that $a(f)\Omega = 0$. Let $f = f_1 \in \mathcal{H}$ be arbitrarily chosen. For $F_n = f_2 \vee \cdots \vee f_{n+1} \in \vee^n\mathcal{H}$, $n \in \mathbf{N}$, and $G_{n+1} = g_1 \vee \cdots \vee g_{n+1} \in \vee^{n+1}\mathcal{H}$ it follows that

$$\begin{aligned}
\langle G_{n+1}, a^*(f_1)F_n \rangle &= \langle g_1 \vee \cdots \vee g_{n+1}, f_1 \vee f_2 \cdots \vee f_{n+1} \rangle \\
&= \sum_{i=1}^{n+1} \langle g_i, f \rangle \cdot \left\langle G_n^{(i)}, F_n \right\rangle \\
&= \left\langle \sum_{i=1}^{n+1} \langle g_i, f \rangle \cdot g_1 \vee \cdots \vee g_{i-1} \cdot g_{i+1} \vee \cdots \vee g_{n+1}, F_n \right\rangle
\end{aligned}$$

where $G_n^{(i)}$ denotes $g_1 \vee \cdots \vee g_{i-1} \vee g_{i+1} \vee \cdots \vee g_{n+1}$ and we have used the formula for the inner product on $\mathcal{F}_\vee(\mathcal{H})$ (see section 1.1). Hence formula (2.6) follows. We have thus shown that the operator defined by (2.5) og (2.6) fulfils that $a(f) \subset a^*(f)^*$. We may therefore choose $\mathcal{D}$ as the dense domain of all the $a(f)$, $f \in \mathcal{H}$. Remark that the mapping $f \mapsto a(f)$ evidently is anti-linear. Since $a^*(f)$ is bounded as an operator from $\vee^n\mathcal{H}$ onto $\vee^{n+1}\mathcal{H}$, so is $a(f) = a^*(f)^*$ as an operator from $\vee^{n+1}\mathcal{H}$ onto $\vee^n\mathcal{H}$, and we have

$$\begin{aligned}
\| a(f) \, |_{\vee^{n+1}\mathcal{H}} \|^2 &= \| a^*(f)|_{\vee^n\mathcal{H}} \|^2 \\
&= (n+1) \cdot \|f\|^2
\end{aligned}$$

by (2.4). Therefore

$$\|a(f)F_n\| \leq \sqrt{n} \cdot \|f\| \cdot \|F_n\| \qquad (2.7)$$

for any $F_n \in \vee^n\mathcal{H}$ and $f \in \mathcal{H}$. Observe that the norm estimate for $a(f)$ is almost the same as the norm estimate for $a^*(f)$, except for a factor $\sqrt{\frac{n}{n+1}}$. We already know that the domain $\mathcal{D}$ for $a(f)$ is dense in $\mathcal{F}_\vee(\mathcal{H})$ and since $a(f) : \vee^n\mathcal{H} \mapsto \vee^{n-1}\mathcal{H}$ it follows that $\mathcal{D}$ is invariant. In fact $\mathcal{D}$ is a set of analytic vectors for all $a(f)$, $f \in \mathcal{H}$, since for any $F = \oplus_{n=0}^{\infty} F_n \in \mathcal{D}$ there exists a $N \in \mathbf{N}$ such that $F_n = 0$ for $n > N$. Therefore $(a(f))^k F = 0$, for

$k > N$. Hence

$$\sum_{k=0}^{\infty} \frac{\left\|(a(f))^k F\right\|}{k!} \cdot t^k = \sum_{k=0}^{N} \frac{\left\|(a(f))^k F\right\|}{k!} \cdot t^k$$

which evidently is finite for all $t \in \mathbf{R}_+$. We call $a(f)$, $f \in \mathcal{H}$, for *annihilation operators*, for obvious reasons. As in the case of the creation operator, we could define $a(f)$ on $\mathcal{D}(N^{1/2})$ which is the maximal domain of $a(f)$ on $\mathcal{F}_\vee(\mathcal{H})$.

We will now verify that $a^*(f)$ and $a(g)$, $f, g \in \mathcal{H}$ fulfil the canonical commutation relations on $\mathcal{D}$. In fact we merely have to prove the claim on product vectors and the vacuum vector, since a finite linear combination of these spans a dense set of $\mathcal{F}_\vee(\mathcal{H})$. Let $n \in \mathbf{N}$ be arbitrarily chosen and consider any $F_n \in \vee^n\mathcal{H}$ of the form $F_n = f_1 \vee \cdots \vee f_n$. Then

$$\begin{aligned} a(g)a^*(f)(f_1 \vee \cdots \vee f_n) &= a(g)(f \vee f_1 \vee \cdots \vee f_n) \\ &= \langle g, f\rangle F_n + \sum_{i=1}^{n} \langle g, f_i\rangle G_n^{(i)} \end{aligned}$$

where $G_n^{(i)} = f \vee f_1 \vee \cdots \vee f_{i-1} \vee f_{i+1} \vee \cdots \vee f_n$, for $i = 1, \ldots, n$. On the other hand

$$\begin{aligned} a^*(f)a(g)(f_1 \vee \cdots \vee f_n) &= a^*(f)\left(\sum_{i=1}^{n} \langle g, f_i\rangle F_n^{(i)}\right) \\ &= \sum_{i=1}^{n} \langle g, f_i\rangle G_n^{(i)} \end{aligned}$$

where $F_n^{(i)} = f_1 \vee \cdots \vee f_{i-1} \vee f_{i+1} \vee \cdots \vee f_n$, for $i = 1, \ldots, n$ and $G_n^{(i)}$ is as above. Then it immediately follows that

$$[a(g), a^*(f)] = \langle g, f\rangle \cdot I$$

on $\vee^n\mathcal{H}$. Moreover, we have $a(g)a^*(f)\Omega = \langle g, f\rangle \Omega$ and $a^*(f)a(g)\Omega = a^*(f)0 = 0$. Hence it follows that

$$[a(g), a^*(f)] = \langle g, f\rangle \cdot I$$

on $\mathcal{D}$. Finally we have

$$[a^*(g), a^*(f)]f_1 \vee \cdots \vee f_n = g \vee f \vee f_1 \vee \cdots \vee f_n - f \vee g \vee f_1 \vee \cdots \vee f_n = 0$$

by the symmetri, hence also $[a(g), a(f)] = 0$ on $\mathcal{D}$. Trivially $[a(g), a(f)]\Omega = 0$ and $[a^*(g), a^*(f)]\Omega = 0$ as well. Thus

$$[a^*(g), a^*(f)] = 0 = [a(g), a(f)]$$

on $\mathcal{D}$. Whence the commutation relations also hold on the maximal domain $\mathcal{D}(N^{1/2})$. We summarize this in the following theorem.

Theorem 9 *The unbounded creation and annihilation operators on the symmetric Fock Hilbert space $\mathcal{F}_\vee(\mathcal{H})$ with domain $\mathcal{D}$ or $\mathcal{D}(N^{1/2})$ as constructed above have $\mathcal{D}$ as a dense set of analytical vectors and fulfil the canonical commutation relations and thereby give a $*$-representation of the $*$-algebra generated by $\{a(f) : f \in \mathcal{H}\}$.*

Proof. A direct consequence of the discussion above. However, most of these statement are also discussed and proved in [B-R 2, section 5.2.1]. ∎

We call this $*$-representation for the Fock representation of the $*$-algebra generated by $\{a(f) : f \in \mathcal{H}\}$. Observe that the vacuum vector Ω is a cyclic vector, since each product vector $f_1 \vee \cdots \vee f_n$ is of the form $a^*(f_1) \cdot \ldots \cdot a^*(f_n)\Omega$.

Define

$$\pi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f)) \tag{2.8}$$

for all $f \in \mathcal{H}$, with domain $\mathcal{D}$. Following [R-S 1, p.209] the operator $\pi(f)$ is called *the Segal field operator* and the mapping $f \to \pi(f)$ is called *the Segal quantization over $\mathcal{H}$*. Hence $a(f)$ and $a^*(f)$ can be recovered from $\pi(f)$ by the formula $a^*(f) = \frac{1}{\sqrt{2}}(\pi(f) - i\pi(if))$ and $a(f) = \frac{1}{\sqrt{2}}(\pi(f) + i\pi(if))$. Accordingly there is a one-to-one correspondence between the two points of view, and we may therefore consider the $*$-algebra generated by $\{\pi(f) : f \in \mathcal{H}\}$. Notice that the mapping $f \mapsto \pi(f)$ is only real linear. From the above it immediately follows that Ω is a cyclic vector, i.e. the linear span of $\{\pi(f_1) \cdot \ldots \cdot \pi(f_n)\Omega : f_1, \ldots, f_n \in \mathcal{H},\ n \in \mathbb{N}\}$ is dense in $\mathcal{F}_\vee(\mathcal{H})$. Moreover, the canonical commutation relations for $a(f)$ and $a(g)$, $f, g \in \mathcal{H}$, gives

$$\begin{aligned}
&[\pi(f), \pi(g)] \\
&\quad = i \cdot Im\,\langle f, g\rangle \cdot I \\
&\quad = i \cdot \sigma(f, g) \cdot I
\end{aligned} \tag{2.9}$$

on $\mathcal{D}$, where $\sigma(f,g) = Im\,\langle f,g\rangle$ is a non-degenerated symplectic bilinear form on $\mathcal{H}$ as a real Hilbert space. The restriction of $\pi(f)$ to $\vee^n\mathcal{H}$ is bounded, since both $a(f)$ and $a^*(f)$ are, in fact

$$\begin{aligned} \|\pi(f)F_n\| &\leq \frac{1}{\sqrt{2}}\left(\|a(f)F_n\| + \|a^*(f)F_n\|\right) \\ &\leq \sqrt{2\cdot(n+1)}\,\|f\|\cdot\|F_n\| \end{aligned} \tag{2.10}$$

Then by induction after $k \in \mathbf{N}$ we get

$$\left\|\pi(f)^k F_n\right\| \leq 2^{\frac{k}{2}}\sqrt{\frac{(n+k)!}{n!}}\cdot\|f\|^k\cdot\|F_n\|$$

where we will use that $\pi(f) : \vee^n\mathcal{H} \to \vee^{n-1}\mathcal{H} \oplus \vee^{n+1}\mathcal{H}$ and the equations (2.4) and (2.7). Assume that $\left\|\pi(f)^k F_n\right\| \leq 2^{\frac{k}{2}}\cdot\sqrt{\frac{(n+k)!}{n!}}\cdot\|f\|^k\cdot\|F_n\|$ then

$$\begin{aligned} \left\|\pi(f)^{k+1}F_n\right\| &\leq \frac{1}{\sqrt{2}}\left(\left\|\pi(f)^k(a(f)F_n)\right\| + \left\|\pi(f)^k(a^*(f)F_n)\right\|\right) \\ &\leq 2^{\frac{k+1}{2}}\cdot\sqrt{\frac{(n+(k+1))!}{n!}}\cdot\|f\|^{k+1}\cdot\|F_n\| \end{aligned}$$

proving the formula for $k+1$. Notice that the formula naturally also holds for $k=0$ and is proven above for $k=1$.

Therefore

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\left\|\pi(f)^k F_n\right\| \leq \sum_{k=0}^{\infty}\frac{(\sqrt{2}\cdot t)^k}{k!}\sqrt{\frac{(n+k)!}{n!}}\cdot\|f\|^k\cdot\|F_n\|$$

is finite for all $t \in \mathbf{R}_+$, by the ratio test. Then each $F_n \in \vee^n\mathcal{H}$ and each $\oplus_{n=0}^{\infty}F_n \in \mathcal{D}$ is an analytic vector for $\pi(f)$. Hence $\pi(f)$ is essentially selfadjoint on $\mathcal{D}$, for each $f \in \mathcal{H}$, by Nelsons analytic vector theorem (see [R-S 2, p. 202]), since $\pi(f)$ is indeed a symmetric operator. Notice that the mapping $f \to \pi(f)$ is strongly continous, by

$$\|(\pi(f)-\pi(g))F\| = \|\pi(f-g)F\| \leq \sqrt{2}\cdot\|f-g\|\cdot\left\|(N+1)^{\frac{1}{2}}F\right\|$$

for $F \in \mathcal{D}$, where N denotes the number operator mentioned above.

The $*$-algebra generated by $\{\pi(f) : f \in \mathcal{H}\}$ is usually denoted the Fock representation of the corresponding abstract $*$-algebra.

We will now use the selfadjoint operators $\overline{\pi(f)}$, the closure of the essentially selfadjoint operators $\pi(f)$, to construct the *Weyl operators* $W(f)$, $f \in \mathcal{H}$, as the unitary operators on $\mathcal{F}_V(\mathcal{H})$ given by

$$W(f) = e^{i\overline{\pi(f)}} \tag{2.11}$$

for $f \in \mathcal{H}$. Henceforth we will just write $\pi(f)$ for the selfadjoint closure $\overline{\pi(f)}$ of $\pi(f)$. Therefore $\pi(f)$ denotes two different operators interchangeably, the one operator is the close extension of the other. If it is not evident from the context to which operator $\pi(f)$ actually refers, we will explicitly mention it. We then get the following proposition, which probably is owing to E. Segal.

Proposition 10 *Let $f, g \in \mathcal{H}$. Then the Weyl operators $W(f)$ and $W(g)$ defined above fulfil*

1) $W(f)W(g) = W(f+g)e^{-\frac{1}{2}\cdot i\cdot\sigma(f,g)}$

2) $W(f)$ *is strongly continous in* f

3) $\langle\Omega, W(f)\Omega\rangle = e^{-\frac{1}{4}\cdot\|f\|^2}$

Proof. This proposition can also be found in [B-R 2, theorem 5.2.4 (p.13) and on p.25]. However, our proof is different from theirs.

1) From equation (2.9) it follows by induction for arbitrary $f, g \in \mathcal{H}$ that

$$\frac{1}{n!}(\pi(f)+\pi(g))^n = \sum_{k+l+2\cdot m=n} \frac{1}{k!}\cdot\pi(f)^k\cdot\frac{1}{l!}\cdot\pi(g)^l\frac{1}{m!}\cdot\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m$$

on $\mathcal{D}$, for any $n \in \mathbf{N}\cup\{0\}$. For $n = 0$, both sides evidently gives I. For $n = 1$, the right hand side gives $\pi(f)+\pi(g)$, which obviously equals the left hand side. Assume that the formula holds for any $n = N \in \mathbf{N}$. This implies that

$$
\begin{aligned}
&\frac{1}{N!}\left(\pi(f)+\pi(g)\right)^{N+1} \\
&= \left(\pi(f)+\pi(g)\right)\cdot\frac{1}{N!}\cdot\left(\pi(f)+\pi(g)\right)^{N} \\
&= \left(\pi(f)+\pi(g)\right)\sum_{k+l+2\cdot m=N}\frac{1}{k!}\pi(f)^k\cdot\frac{1}{l!}\pi(g)^l\cdot\frac{1}{m!}\left(-\frac{1}{2}i\sigma(f,g)\right)^m \\
&= \sum_{k+l+2\cdot m=N}\left(\frac{1}{k!}\cdot\pi(f)^{k+1}\cdot\frac{1}{l!}\cdot\pi(g)^l\cdot\frac{1}{m!}\cdot\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m\right. \\
&\quad +\frac{1}{k!}\cdot\pi(f)^k\cdot\frac{1}{l!}\cdot\pi(g)^{l+1}\cdot\frac{1}{m!}\cdot\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m \\
&\quad \left.+\ k\cdot\frac{1}{k!}\cdot\pi(f)^{k-1}\cdot\frac{1}{l!}\cdot\pi(g)^l\cdot\frac{1}{m!}\cdot 2\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^{m+1}\right)
\end{aligned}
$$

where we have used equation (2.9) together with $\sigma(g,f)=-\sigma(f,g)$, whereby

$$
\left[\pi(g),\pi(f)^k\right]=k\cdot 2\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)\pi(f)^{k-1}
$$

on $\mathcal{D}$, for every $k\in\mathbf{N}$. Hence we get

$$
\begin{aligned}
&\frac{1}{N!}\left(\pi(f)+\pi(g)\right)^{N+1} \\
&= \sum_{k+l+2\cdot m=N+1}\left(k\cdot\frac{1}{k!}\cdot\pi(f)^k\cdot\frac{1}{l!}\cdot\pi(g)^l\cdot\frac{1}{m!}\cdot\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m\right. \\
&\quad +l\cdot\frac{1}{k!}\cdot\pi(f)^k\cdot\frac{1}{l!}\cdot\pi(g)^l\cdot\frac{1}{m!}\cdot\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m \\
&\quad \left.+\ 2m\cdot\frac{1}{k!}\cdot\pi(f)^k\cdot\frac{1}{l!}\cdot\pi(g)^l\cdot\frac{1}{m!}\cdot\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m\right) \\
&= (N+1)\sum_{k+l+2\cdot m=N+1}\frac{1}{k!}\pi(f)^k\cdot\frac{1}{l!}\pi(g)^l\cdot\frac{1}{m!}\left(-\frac{1}{2}\cdot i\cdot\sigma(f,g)\right)^m
\end{aligned}
$$

from which the desired formula immediately follows for $n=N+1$, by dividing by $N+1$, whereby the induction is completed. It follows by Cauchy-Schwarz's inequality that the double series

$$
\sum_{k,l=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{l!}\left\langle\pi(g)^lG,\pi(f)^kF\right\rangle
$$

converges absolutely for any $F, G \in \mathcal{D}$. Now the results given above implies that

$$\begin{aligned}
&\langle W(f+g)G, F\rangle \\
&= \left\langle \sum_{n=0}^{\infty} \frac{1}{n!}\pi(f+g)^n G, F\right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle \frac{1}{n!}\left(\pi(f)+\pi(g)\right)^n G, F\right\rangle \\
&= \sum_{n=0}^{\infty} \sum_{k+l+2\cdot m=n} \left\langle \frac{1}{l!}\pi(g)^l G, \frac{1}{k!}\pi(f)^k F\right\rangle \cdot \frac{1}{m!}\left(-\frac{1}{2}i\sigma(f,g)\right)^m \\
&= \left\langle \sum_{l=0}^{\infty} \frac{1}{l!}\pi(g)^l G, \sum_{k=0}^{\infty} \frac{1}{k!}\pi(f)^k F\right\rangle \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{1}{2}i\sigma(f,g)\right)^m \\
&= \langle W(g)G, W(f)F\rangle \cdot e^{-\frac{1}{2}\cdot i\cdot\sigma(f,g)} \\
&= \left\langle W(f)W(g)\cdot e^{-\frac{1}{2}\cdot i\cdot\sigma(f,g)} G, F\right\rangle
\end{aligned}$$

for all $F, G \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $\mathcal{H}$ it follows that

$$W(f)W(g) = W(f+g)\cdot e^{+\frac{1}{2}\cdot i\cdot\sigma(f,g)}$$

for any $f, g \in \mathcal{H}$, proving 1). Notice that we, as a corollary have obtained

$$W(f)W(g) = W(g)W(f)\cdot e^{i\cdot\sigma(f,g)}$$

on $\mathcal{H}$ and that $W(-f)W(f) = W(0) = I$, for all $f, g \in \mathcal{H}$.

2) For any $F \in \mathcal{D}$ there exists an $n \in \mathbf{N}$ such that $F_n = 0$, for $n > N$, where $F = \oplus_{n=0}^{\infty} F_n$. The previously derived norm estimate

$$\left\|\pi(f)^k F_n\right\| \leq 2^{\frac{k}{2}} \cdot \sqrt{\frac{(n+k)!}{n!}} \cdot \|f\|^k \cdot \|F_n\|$$

then gives that $\left\|\pi(f)^k F_n\right\| \leq 2^{\frac{k}{2}} \cdot \sqrt{\frac{(N+k)!}{N!}} \cdot \|f\|^k \cdot \|F\|$, for any $f \in \mathcal{H}$. For any $f \in \mathcal{H}$ and any $F \in \mathcal{D}$ we therefore have

$$\begin{aligned}
\|(W(f)-1)F\| &= \left\|\sum_{k=1}^{\infty} \frac{1}{k!}\pi(f)^k F\right\| \\
&\leq \sum_{k=1}^{\infty} \frac{1}{k!} \cdot 2^{\frac{k}{2}} \cdot \sqrt{\frac{(N+k)!}{N!}} \cdot \|f\|^k \cdot \|F\|
\end{aligned}$$

which converges, by the ratio test, with a limit less than the convergent series $\sum_{k=1}^{\infty} \frac{1}{k!} \cdot 2^{\frac{k}{2}} \cdot \sqrt{\frac{(N+k)!}{N!}}$ times $\|f\| \cdot \|F\|$, for f in the unit ball. Hence $\|(W(f)-1)F\| \to 0$, as $f \to 0$, i.e. $W(f)$ is strongly continous at $f=0$ on $\mathcal{D}$, and then on all of $\mathcal{H}$.

For arbitrary $f, g \in \mathcal{H}$, we obtain the following

$$\begin{aligned}
&\|(W(f) - W(g))F\| \\
&\quad = \|W(g)(W(-g)W(f) - 1)F\| \\
&\quad \leq \left\| \left(W(f-g) \cdot e^{-\frac{1}{2} \cdot i \cdot \sigma(f,g)} - 1 \right) F \right\| \\
&\quad \leq \|(W(f-g) - 1)F\| + \left| 1 - e^{+\frac{1}{2} \cdot i \cdot \sigma(f,g)} \right| \cdot \|F\| \\
&\quad \to 0 \quad \text{as } g \to f
\end{aligned}$$

by use of **1**), the strong continuity of $W(f)$ at zero, the continuity of $\sigma(f,g) = Im\,\langle f, g\rangle$ and the fact that $\sigma(f,f) = 0$. Hence $W(f)$ is strongly continous in $f \in \mathcal{H}$, proving **2**).

3) We first notice that $[a(f), \pi(f)] = \frac{1}{\sqrt{2}} \|f\|^2 \cdot I$, by equation (2.1) and (2.8). This implies that

$$\left[a(f), \pi(f)^k\right] = k \cdot \frac{1}{\sqrt{2}} \cdot \|f\|^2 \cdot \pi(f)^{k-1} \tag{2.12}$$

for all $k \in \mathbf{N}$, on $\mathcal{D}$. For $n \geq 2$ we get

$$\begin{aligned}
\langle \Omega, \pi(f)^n \Omega \rangle &= \left\langle \pi(f)\Omega, \pi(f)^{n-1}\Omega \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle a^*(f)\Omega, \pi(f)^{n-1}\Omega \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle \Omega, a(f)\pi(f)^{n-1}\Omega \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle \Omega, (n-1)\frac{1}{\sqrt{2}} \|f\|^2 \cdot \pi(f)^{n-2}\Omega \right\rangle \\
&= \frac{n-1}{2} \|f\|^2 \left\langle \Omega, \pi(f)^{n-2}\Omega \right\rangle
\end{aligned} \tag{2.13}$$

where we have used equation (2.12) for $k = n-1$.

Observe now that $\langle \Omega, \pi(f)^0 \Omega \rangle = \langle \Omega, \Omega \rangle = 1$, $\langle \Omega, \pi(f)\Omega \rangle = \left\langle \Omega, \frac{1}{\sqrt{2}} f \right\rangle = 0$, $\langle \Omega, \pi(f)^2 \Omega \rangle = \|\pi(f)\Omega\|^2 = \frac{1}{2} \|f\|^2$ and $\langle \Omega, \pi(f)^3 \Omega \rangle = \left\langle f, \|f\|^2 \Omega + f \vee f \right\rangle =$

0. So iteration by use of equation (2.13) and the recently derived properties gives us that

$$\langle \Omega, \pi(f)^n \Omega \rangle = \begin{cases} 0 & , \text{ for } n \text{ odd} \\ \frac{(n-1)(n-3)\cdot \ldots \cdot 1}{2^{n/2}} \cdot \|f\|^n & , \text{ for } n \text{ even} \end{cases}$$

Rewriting $(2n-1)(2n-3)\cdot \ldots \cdot 1$ as $\frac{(2n)!}{2^n \cdot n!}$ gives us that

$$\left\langle \Omega, \pi(f)^{2n+1} \Omega \right\rangle = 0$$

and

$$\left\langle \Omega, \pi(f)^{2n} \Omega \right\rangle = \frac{(2n)!}{2^{2n} \cdot n!} \cdot \|f\|^{2n}$$

for $n \in \mathbf{N} \cup \{0\}$. Since Ω is an analytic vector for $\pi(f)$, we may use the following expansion

$$\begin{aligned} \langle \Omega, W(f)\Omega \rangle &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \Omega, \pi(f)^n \Omega \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{(2n)!}{2^{2n} \cdot n!} \cdot \|f\|^{2n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{4} \cdot \|f\|^2\right)^n \\ &= e^{-\frac{1}{4} \cdot \|f\|^2} \end{aligned}$$

by use of the formulas derived above, proving **3**). We notice that claim **2**) also follows from **3**) by direct calculation. ∎

The concrete C^*-algebra, generated by the Weyl operators $W(f)$, $f \in \mathcal{H}$, is called the *Fock representation of the CCR-algebra.* For completeness reasons, we will briefly discuss the (abstract) CCR-algebra below, though it is of no need for the argumentation to come. We will, however, state some propositions first.

Proposition 11 *The Fock space $*$-representations $a(f)$ and $\pi(f)$ are irreducible, in the sense that any bounded operator T on $\mathcal{F}_\vee(\mathcal{H})$ commuting with, the $*$-algebra generated by $a(f)$ and $\pi(f)$ respectively is trivial i.e. T is a scalar multiple of the unit operator.*

Proof. Since $a(f) = \frac{1}{\sqrt{2}}(\pi(f) + i\pi(if))$ and $a^*(f) = \frac{1}{\sqrt{2}}(\pi(f) - i\pi(if))$, any operator T commuting with $\pi(f)$, $f \in \mathcal{H}$, commutes with $a(f)$ and $a^*(f)$, and conversely. Therefore we only need to consider one of the statements, since they imply each other. Let T commute with $a(f)$ and $a^*(f)$ on $\mathcal{D}$, i.e. $T\mathcal{D} \subset \mathcal{D}$ and $[T, a(f)] = [T, a^*(f)] = 0$, on $\mathcal{D}$. Then the rest of the proof is quite simmilar to that of theorem 1 ∎

Corollary 12 *The Fock representation $W(f)$, $f \in \mathcal{H}$, of the CCR-algebra is irreducible.*

Proof. For a proof, see [R-S 2, p.232]. It seem that the argument of the proof in [B-R 2, p.13] fails. ∎

Analogously to the treatment of the CAR-algebra (see section 1.2 and 1.4), we may consider $\mathcal{H}$ as a real Hilbert space, equipped with a non-degenerated symplectic bilinear form σ, i.e. $\sigma(f,g) = -\sigma(g,f)$, for all $f,g \in \mathcal{H}$, and if $f \in \mathcal{H} \setminus \{0\}$ there exists a $g \in \mathcal{H}$ such that $\sigma(f,g) \neq 0$. Notice that the σ previously defined in (2.9), is an example of such a form. We are now ready to define *the (abstract) CCR-algebra* (following [B-R 2, p.20]).

Definition 13 *Let $\mathcal{H}$ be a real Hilbert space equipped with a non-degenerated symplectic bilinear form σ, then the CCR-algebra is the C^*-algebra generated by non-zero elements $W(f)$, $f \in \mathcal{H}$, satisfying that $W(-f) = W(f)^*$ and that $W(f)W(g) = e^{-\frac{1}{2}\cdot i\cdot\sigma(f,g)} \cdot W(f+g)$ for all $f,g \in \mathcal{H}$.*

Notice that the previously constructed concrete C^*-algebra, generated by the Weyl operators, fulfils the condition in the definition of being a CCR-algebra. Observe that we indicate some uniqueness by use of "the" instead of "a" in "the CCR-algebra". This is justified in the following theorem.

Theorem 14 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two CCR-algebras, then there exists a unique $*$-isomorphism $\alpha : \mathcal{A}_1 \mapsto \mathcal{A}_2$ mapping the generators $W_1(f)$ of $\mathcal{A}_1$ into the generators $W_2(f)$ of $\mathcal{A}_2$, for all $f \in \mathcal{H}$. Thus the CCR-algebra is unique, up to $*$-isomorhism. Moreover, $W(0) = 1$ and $W(f)$ is unitary, for all $f \in \mathcal{H}$.*

Furthermore, if S is a real linear invertible operator on $\mathcal{H}$, leaving σ invariant, i.e. $\sigma(Sf, Sg) = \sigma(f,g)$ for all $f, g \in \mathcal{H}$, then there exists a unique $*$-automorphism α on the CCR-algebra, such that $\alpha(W(f)) = W(S^{-1}f)$, for all $f \in \mathcal{H}$.

Proof. We will not prove this theorem, except for the last part, since we won't need it throughout this thesis. For a detailed proof of the entire theorem consult [B-R 2, p. 20-22.]. However, the last part follows directly from the first part, with $W_1(f) = W(f)$ and $W_2(f) = W(S^{-1}f)$, $f \in \mathcal{H}$. Such real, linear, invertible operators S on $\mathcal{H}$ leaving σ invariant, are usually called Boguliubov transformations.

The last part can be shown directly, though. We only have to show that $W_2(f) = W(S^{-1}f)$ fulfils the Weyl form of the canonical anti-commutation relations, given in 1) of proposition 10. We get $W(S^{-1}f)W(S^{-1}g) = W(S^{-1}(f+g))\cdot e^{-\frac{1}{2}i\sigma(S^{-1}f, S^{-1}g)} = W(S^{-1}(f+g))\cdot e^{-\frac{1}{2}i\sigma(f,g)}$ that is $W_2(f)$, $f \in \mathcal{H}$, form a CCR-algebra by definition 13, so the mapping $W(f) \mapsto W(S^{-1}f)$ defines a unique $*$-morphism. ∎

The generalization of the Fock representation of the CCR-algebra above is somehow very slight. Let σ be a non-degenerated symplectic form on a real Hilbert space $\mathcal{H}_r$, for example, $\mathcal{H}$ considered as a real Hilbert space. Then there exists a real linear operator J on $\mathcal{H}_r$ such that $J^2 = -I$ and $\sigma(Jf, g) = -\sigma(f, Jg)$ for all $f, g \in \mathcal{H}_r$. In fact, if $\{u_k, v_k\}_{k \in \mathbf{N}}$ is a symplectic basis for $\mathcal{H}_r$, i.e. $\sigma(u_k, u_l) = \sigma(v_k, v_l) = 0$ and $\sigma(u_k, v_l) = \delta_{k-l}$, for all $k, l \in \mathbf{N}$, then we put $Ju_k = v_k$ and $Jv_k = -u_k$. Extension of J by (real) linearity and continuity gives us an operator with the above properties on $\mathcal{H}_r$ (one can always choose an symplectic basis for an even or infinite dimensional space, see for example [Arn, p. 220]). Such an operator J introduce a complex structure on $\mathcal{H}_r$, and reflects the complex structure of a complex Hilbert space $\mathcal{H}$. The correspondence between $\mathcal{H}$ and $\mathcal{H}_r$ is then given by $e_k \leftrightarrow u_k$ and $i \cdot e_k \leftrightarrow v_k = Ju_k$, where $\{e_k = u_k\}_{k \in \mathbf{N}}$ is a basis of $\mathcal{H}$, and $\langle f, g\rangle = \sigma(f, Jg) + i\sigma(f,g)$ is an inner product on $\mathcal{H}$. Observe that $\sigma(f,g) = Im\,\langle f, g\rangle$, for all $f, g \in \mathcal{H}$, and $\mathcal{H}_r$ is $\mathcal{H}$ considered as a real Hilbert space.

We now turn to a construction called second quantization in $\mathcal{F}_\vee(\mathcal{H})$, which we will treat in the next section.

2.2 The second quantization in $\mathcal{F}_{\vee}(\mathcal{H})$.

In this section we will describe the second quantization, based on the explicit Fock Hilbert space construction in the symmetric case, given in section 1.1, and on the Fock $*$-representation of the $*$-algebra generated by $\{a(f) : f \in \mathcal{H}\}$, given in section 2.1, following the pattern outlined in the anti-symmetric case, given in section 1.3. This section will be used frequently in the construction of the socalled metaplectic representation, to be treated in the next section.

As mentioned in section 1.3, the basic idea of second quantization does go back to V. Fock ([Fo]) in 1932. However, it was J.M Cook ([Co]) who in 1953 made the construction in details. The construction is a method of lifting one-particle operators on a Hilbert space to many-particle operators on the Fock Hilbert space, whenever it is possible (in the symmetric case, one uses the symmetric Fock Hilbert space). It gave a nice mathematical frame for handling many problems in quantum physics with infinitely many degrees of freedom. The idea of this method, called second quantization, has been somewhat generalized by Lundberg in [Lu 2]. Our treatment of second quantization is somewhat different from those given by Cook in [Co] and by Bratteli and Robinson in [B-R 2].

Similar to section 1.3 one define a strongly continuous one-parameter unitary group of operators $U(A)_n$ on $\vee^n\mathcal{H}$, for any bounded skew-selfadjoint operator A, given by $U(A)_0\Omega = \Omega$ and

$$U(A)_n(f_1 \vee \ldots \vee f_n) = e^A f_1 \vee \ldots \vee e^A f_n \tag{2.14}$$

on product vectors $f_1 \vee \ldots \vee f_n \in \vee^n\mathcal{H}$. The infinetesimal generators $dU(A)_n = \frac{d}{dt}\Big|_{t=0} U(tA)_n$ on its domain fulfils

$$U(tA)_n = e^{t \cdot dU(A)_n} \tag{2.15}$$

and become

$$dU(A)_n F_n = \sum_{i=1}^{n} f_1 \vee \ldots \vee f_{i-1} \vee Af_i \vee f_{i+1} \vee \ldots \vee f_n \tag{2.16}$$

on product vectors. Moreover, $dU(A)_0\Omega = 0$.

Similar to our preceding notational conventions, we let $\mathcal{D}$ denote the algebraic direct sum $\oplus_{alg} \vee^n \mathcal{H}$. It then follows that each $dU(A)_n$ leaves $\vee^n \mathcal{H}$ invariant and that $dU(A) = \oplus_{n=0}^{\infty} dU(A)_n$, is well-defined on $\mathcal{D}$. Observe that $dU(A)$ is skew-symmetric with a dense invariant domain $\mathcal{D}$.

Consider now possible unbounded skew-selfadjoint operators A. We then define $dU(A)_n$ directly by $dU(A)_0\Omega = 0$ and by (2.16) on product vectors in $\vee^n \mathcal{H}$, such that each one-particle vector $f_1, \ldots, f_n$ belongs to the domain $\mathcal{D}(A)$ of A. We denote this set of product vectors by $\mathcal{D}(A)^{\vee n}$, for each $n \in \mathbf{N} \cup \{0\}$. Then is $dU(A)_n$ well-defined on $\mathcal{D}(A)^{\vee n}$. Put $dU(A) = \oplus_{n=0}^{\infty} dU(A)_n$ on the algebraic direct sum $\oplus_{alg} \mathcal{D}(A)^{\vee n}$.

Completely analogue to the anti-symmetric case one shows that

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!} \left\| dU(A)^k (f_1 \vee \ldots \vee f_n) \right\| &\leq \sum_{k=0}^{\infty} \frac{(n \cdot M)^k}{k!} \cdot \|f_1 \otimes \cdots \otimes f_n\| \cdot (n!)^{\frac{1}{2}} \\
&= (n!)^{\frac{1}{2}} \cdot e^{n \cdot M} \cdot \|f_1 \otimes \cdots \otimes f_n\| \\
&< \infty
\end{aligned}
$$

on any product vector of analytic vectors for A. This means that the set $\mathcal{D}_A$ of finite linear combinations of Ω and n-folded tensor products $f_1 \vee \ldots \vee f_n$, $n \in \mathbf{N}$, such that each one-particle vector f_j is an analytic vector for A, is a set of analytic vectors for $dU(A)$.

We observe that $\mathcal{D}_A$ is a dense set, since the analytic vectors for A in $\mathcal{D}(A)$ form a dense set in $\mathcal{H}$. Then Nelson's theorem (transformed to essentially skew-selfadjoint operators) state that the operator $dU(A)$ is essentially skew-selfadjoint (this is shown by other arguments in [R-S 1, p. 302]). Hence, the closure of $dU(A)$, which we also denote by $dU(A)$, is skew-selfadjoint and generates a strongly continous one-parameter unitary group by (a transformation of) Stone's theorem. We denote this strongly continous unitary one-parameter group by $U(tA)$. Notice that in the case of A bounded, the above M may be chosen equal to $\|A\|$ and $\mathcal{D}_A$ becomes all of $\mathcal{D}$ (all vectors in $\mathcal{H}$ is an analytic vector for A bounded).

The above mapping $A \to dU(A)$, which maps skew-selfadjoint operators on $\mathcal{H}$ into skew-selfadjoint operators on $\mathcal{F}_{\vee}(\mathcal{H})$ is called *the second quantization* mapping. Remark that we will use the commutator $[B, C]$ to denote $BC - CB$, even for unbounded operators, of course one has to specify on which domain the commutator is defined. In our case the two operators have at least $\mathcal{D}$ as a common invariant domain, so the commutator is well-defined on $\mathcal{D}$. However, one have to be very careful, since a vanishing commutator on a dense set does not necessarily mean that the unbounded operators do commute (see for example [R-S 1, p. 272-273 and p. 306-307]).

Theorem 15 *The second quantization mapping, $A \to dU(A)$, on skew-selfadjoint operators A in $\mathcal{H}$ fulfils*

1) $U(tA)a(f)U(-tA) = a(e^{tA}f)$ *on $\mathcal{D}$, for analytic vectors f for A and $t \in \mathbf{R}$.*

2) $\overline{[dU(A), a^*(f)]} = a^*(Af)$ *for all $f \in \mathcal{D}(A)$.*

3) $dU([A,B]) = \overline{[dU(A), dU(B)]}$, *at least for A and B bounded and skew-selfadjoint on $\mathcal{H}$.*

Proof. **1)** The proof is similar to that of theorem 2 part 1), since for each analytic vector f_j for A is also $e^{-tA}f_j$ an analytic vector for A. In addition one have to controle the domain, which is easily done.

2) Follows by considerations completely analogue to those of theorem 2 part 2).

3) In this case the proof of theorem 2 part 3) fails, however the following proof could as well have been used in the earlier case, theorem 2 part 3), with obvious modifications. For A and B bounded and skew-selfadjoint on $\mathcal{H}$ is

$$\begin{aligned} & dU(A)dU(B)(f_1 \vee \ldots \vee f_n) \\ &= \sum_{j=1}^{n}\sum_{\substack{i=1 \\ j\neq 1}}^{n} f_1 \vee \ldots \vee f_{i-1} \vee Bf_i \vee f_{i+1} \vee \ldots \vee f_{j-1} \vee Af_j \vee f_{j+1} \vee \ldots \vee f_n \\ &\quad + \sum_{i=1}^{n} f_1 \vee \ldots \vee ABf_i \vee \ldots f_n \end{aligned}$$

and

$$\begin{aligned} & [dU(A), dU(B)](f_1 \vee \ldots \vee f_n) \\ &= \sum_{i=1}^{n} f_1 \vee \ldots \vee [A,B]f_i \vee \ldots \vee f_n \\ &= dU([A,B])(f_1 \vee \ldots \vee f_n) \end{aligned}$$

where $f_1 \vee \ldots \vee f_n \in \vee^n \mathcal{H}$ is chosen arbitrarily, hence

$$[dU(A), dU(B)] = dU([A, B])$$

on $\mathcal{D}$. ∎

Later on in section 2.3, we will use this result, but only for bounded operators A and B. Notice that part 3) of theorem 15 gives that the second quantization mapping $A \to dU(A)$, is a Lie algebra homomorphism.

A standard example arises for $A = i \cdot I$, then $dU(A)(f_1 \vee \ldots \vee f_n) = i \cdot n \cdot (f_1 \vee \ldots \vee f_n)$, we call $N = -idU(A)$ for the number operator on $\mathcal{F}_\vee(\mathcal{H})$, with dense domain $\mathcal{D}(N) = \{F = \oplus_{n=0}^{\infty} F_n \in \mathcal{F}_\vee(\mathcal{H}) : \sum_{n=0}^{\infty} n^2 \cdot \|F_n\|^2 < \infty\}$ So $N(\oplus_{n=0}^{\infty} F_n) = \oplus_{n=0}^{\infty}(n \cdot F_n)$ for each $\oplus_{n=0}^{\infty} F_n \in \mathcal{D}(N)$. Since N is given in its spectral representation, it is evidently selfadjoint, by von Neumann's theorem (see [R-S 1, p. 275]), as it should be, since $dU(A)$ is skew-selfadjoint.

In the next section, we will use the method of second quantization to construct a metapletic representation, i.e. a representation of the restricted metaplectic group.

2.3 The infinite dimensional metaplectic representation.

In this section we introduce the symplectic group and construct the so-called metaplectic representation of the restricted symplectic group on a Lie algebra level, by ideas similar to that of second quantization. We will follow a strategy parallel to that outlined in section 1.4 in discussing the orthogonal group and the spin representation, and use section 1.1 intensively.

Many authors have studied these subjects, however we will only mention a few [Lu 2], [Ya], [P-S] and [Sh]. We follow [Lu 2] where some of the results are present, but without proofs.

We continue with the earlier notation, exept that we will use $\mathcal{H}$ to denote both a complex Hilbert space, as usually, and the corresponding real Hilbert space (if necessary we put an r-index on $\mathcal{H}$, in the last case, to avoid confusion). Moreover σ denotes the explicit non-degenerated symplectic form, $\sigma(\cdot,\cdot) = Im\,\langle\cdot,\cdot\rangle$, on $\mathcal{H}_r$.

We define *the symplectic group* $Sp(\mathcal{H})$ as the group consisting of those real linear invertible operators $S : \mathcal{H} \to \mathcal{H}$ where $\sigma(Sf, Sg) = \sigma(f,g)$, for all $f, g \in \mathcal{H}$. This definition is analogous to the definition of the orthogonal group given in section 1.4. As in that case we may introduce a complex structure in the real Hilbert space $\mathcal{H}_r$ (see also the discussion in the end of section 2.1).

If $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis of $\mathcal{H}$ and we let u_k and v_k denote e_k and $i\cdot e_k$, respectively, then $\{u_k, v_k\}_{k\in\mathbf{N}}$ form an orthonormal basis for $\mathcal{H}_r$ with respect to $\tau(\cdot,\cdot) = Re\,\langle\cdot,\cdot\rangle$ and the complex structure of $\mathcal{H}$ is then reflected by a complex structure operator J on $\mathcal{H}_r$, which is the real linear bounded operator, discussed in the end of section 2.1, given by $Ju_k = v_k$ and $Jv_k = -u_k$, and fulfilling that $J^2 = -I$ and $\sigma(Jf, g) = -\sigma(f, Jg)$, for all $f, g \in \mathcal{H}_r$, i.e. $J^\sigma = -J$, where J^σ denote the transpose of J relative to $\sigma(\cdot,\cdot)$.

Because of the unique correspondence between $\mathcal{H}$ and $\mathcal{H}_r$, given by $e_k \leftrightarrow u_k$ and $i\cdot e_k \leftrightarrow v_k = Ju_k$, we will not emphasize which space we consider and therefore drop the r-index, unless confusion may arise. It now follows that any real linear mapping $S : \mathcal{H} \to \mathcal{H}$ can be split into a sum of a complex linear operator S_1 and a complex anti-linear operator S_2 as $S = S_1 + S_2$, by putting $S_1 = \frac{1}{2}(S - JSJ)$ and $S_2 = \frac{1}{2}(S + JSJ)$, as in the case of the orthogonal group. Notice that $[S_1, J] = 0$ and $[S_2, J]_+ = 0$. As in the earlier case, the subscripts 1 and 2 will refer to this splitting in the following. The adjoint S_1^* of S_1 is the usual adjoint of a complex linear operator, given by $\langle f, S_1 g\rangle = \langle S_1^* f, g\rangle$, for all $f, g \in \mathcal{H}$. In contrast to this, the adjoint S_2^* of the anti-linear operator S_2 is given by $\langle f, S_2 g\rangle = \langle g, S_2^* f\rangle$, for all $f, g \in \mathcal{H}$, as the usual adjoint of a complex linear operator S_2 from $\mathcal{H}$ to the conjugated Hilbert space $\mathcal{H}^*$. So the adjoint operation means different things, corresponding to the subscript of the operator it is applied to, even though we use an asterisk to denote it in both cases.

Since $S^\sigma S$ is the identity on $\mathcal{H}_r$, for any $S \in \mathcal{S}p(\mathcal{H})$, we get that $S^{-1} = S^\sigma$. Direct calculation gives

$$\begin{aligned}\sigma(S^\sigma f, g) &= \sigma(f, S_1 g) + \sigma(f, S_2 g)\\ &= \sigma(S_1^* f, g) + \sigma(g, S_2^* f)\\ &= \sigma\left((S_1^* - S_2^*)f, g\right)\end{aligned}$$

for all $f, g \in \mathcal{H}$, where we have used the skew-symmetry of the form $\sigma(\cdot,\cdot)$. Then it follows that

$$S^{-1} = S^\sigma = S_1^* - S_2^*$$

Notice that we then have that $S_1^* = (S^\sigma)_1$, but $S_2^* = -(S^\sigma)_2$. Moreover

$$\begin{aligned}I &= S^{-1}S = S^\sigma S\\ &= (S_1^* - S_2^*)(S_1 + S_2)\\ &= (S_1^* S_1 - S_2^* S_2) + (S_1^* S_2 - S_2^* S_1)\end{aligned}$$

where each term is complex linear in the first bracket and complex anti-linear in the second bracket, so

$$S_1^* S_1 - S_2^* S_2 = I \tag{2.17}$$

and

$$S_1^* S_2 - S_2^* S_1 = 0 \tag{2.18}$$

Naturally $S_1 S_1^* - S_2 S_2^* = I$ and $S_2 S_1^* - S_1 S_2^* = 0$, in a similar manner. We will call S_1 and S_2 fulfilling these criteria for *Bogoliubov transformations.* From the last part of theorem 14 we know that such Bogoliubov transformations determinates unique automorphisms on the CCR-algebra, by $W(f) \to W(S^{-1}f)$, for all $f \in \mathcal{H}$, where $W(f)$ denote the Weyl operators (we use the word automorphisms synonymously with $*$-automorphisms). We will give an alternative proof, in our case, in a form we will use later on.

Since $[\pi(S^{-1}f), \pi(S^{-1}g)] = \sigma(S^{-1}f, S^{-1}g) \cdot I = \sigma(f,g) \cdot I$, we observe that the commutation relations, on this form, are invariant under the action $\pi(f) \to \pi_S(f) = \pi(S^{-1}f)$, $S \in \mathcal{S}p(\mathcal{H})$. By the construction of the Weyl operators it follows, once more, that the Weyl form of the canonical commutation relations are left invariant. As in the case of the orthogonal group, there is a natural question to ask; for which $S \in \mathcal{S}p(\mathcal{H})$ is the automorphism given by $\pi(f) \to \pi_S(f) = \pi(S^{-1}f)$, and then the automorphism given by $W(f) \to W_S(f) = W(S^{-1}f)$, unitary implementable (in the Fock representation), i.e. for which $S \in \mathcal{S}p(\mathcal{H})$ does there exist a unitary operator $U(S)$ on $\mathcal{F}_\vee(\mathcal{H})$ such that $\pi_S(f) = U(S)^*\pi(f)U(S)$, for all $f \in \mathcal{H}$, where $\pi_S(f) = \pi(S^{-1}f)$.

Before answering this question we have to define a subgroup $\mathcal{S}p_2(\mathcal{H})$, called *the restricted symplectic group*, of $\mathcal{S}p(\mathcal{H})$.

$$\mathcal{S}p_2(\mathcal{H}) = \{S \in \mathcal{S}p(\mathcal{H}) : S_2 \in \mathbf{L}_2(\mathcal{H})\}$$

where $\mathbf{L}_2(\mathcal{H})$, as before denotes the Hilbert-Schmidt operators on $\mathcal{H}$. It follows that $\mathcal{S}p_2(\mathcal{H})$ really is a group, since $(S^{-1})_2 = -S_2^*$ and $(ST)_2 = S_1T_2 + S_2T_1$ are Hilbert-Schmidt operators, for all $S, T \in \mathcal{S}p_2(\mathcal{H})$.

The restricted symplectic group can be given the structure of a topological group in several different ways, in complete analogy with the restricted orthogonal group. The verification in the case of the strongest topology, which is the topology given by the uniform topology on the linear part and the Hilbert-Schmidt topology on the anti-linear part, $\|S\|_{\mathcal{S}p_2} = \|S_1\| + \|S_2\|_{HS}$ is the same as in the case of the restricted orthogonal group, given in section 1.4 (except that we should write a $\mathcal{S}p_2$-index on the norm symbol instead of the $\mathcal{O}_2$-index). However, the restricted symplectic group is probably connected, but we are not aware of any proof of this.

The choice of topology on $\mathcal{S}p_2(\mathcal{H})$ determine the Lie algebra of $\mathcal{S}p_2(\mathcal{H})$. Our choice of "pre-Lie-algebra" is the simplest one, we choose

$$sp_2(\mathcal{H}) = \{A \in \mathbf{L}_r(\mathcal{H}) : A^\sigma = -A, A_2 \in \mathbf{L}_2(\mathcal{H})\}$$

where $\mathbf{L}_r(\mathcal{H})$ denote the bounded real linear operators on $\mathcal{H}$. The phrase "pre-Lie-algebra" means that we in some applications have to enlarge the "pre-Lie-algebra" to allow operators with unbounded linear part. The demand $A^\sigma = -A$ means that $A_1^* = -A$, and $A_2^* = A_2$, where the adjoint is taken in their respective senses.

In the following we will consider some particular S in $\mathcal{S}p_2(\mathcal{H})$ in a neighbourhood of the identity, generated from $sp_2(\mathcal{H})$ by the exponential mapping.

We return to the question, for which $S \in \mathcal{S}p(\mathcal{H})$ are the automorphisms, defined by $W(f) \to W_S(f) = W(S^{-1}f)$, unitarily implementable in the Fock representation, or equivalently, is there a unitary operator $U(S)$ on $\mathcal{F}_V(\mathcal{H})$ such that $\pi_S(f) = U(S)^{-1}\pi(f)U(S)$, for all $f \in \mathcal{H}$, where $\pi_S(f) = \pi(S^{-1}f)$. In fact this question has already been answered by Shale in 1962, as stated in the following theorem.

Theorem 16 *A unitary operator $U(S)$, which implements the automorphism $\pi(f) \to \pi_S(f)$ exists if and only if $S \in \mathcal{S}p_2(\mathcal{H})$. Moreover, the operator $U(S)$ is unique up to a phase of modulus one.*

Proof. A proof can be found in [Sh], however, we make a construction of $U(S)$, for particular S in a neighbourhood of the identity in $\mathcal{S}p_2(\mathcal{H})$ below. ∎

In the following we will construct the unitary operator $U(S)$ for S in a neighbourhood of the identity in $\mathcal{S}p_2(\mathcal{H})$ in such a way that $U(\cdot)$ lifts one-parameter groups (near the identity) into one-parameter groups, whereby the phase is determined. We emphasize that this particular choise of fixing the phase is different from Shale's choise (see [Sh]). The following construction is analogue to that in the corresponding anti-symmetric case, however, there are some minner but essential differences. Below, we bring the details, pointing out the difference mentioned above.

Since

$$U(TS)^{-1}\pi(f)U(TS) = \left(U(T)U(S)\right)^{-1} \pi(f) \left(U(T)U(S)\right)$$

for $S, T \in \mathcal{S}p_2(\mathcal{H})$, by a direct calculation, and the $*$-algebra generated by $\{\pi(f) : f \in \mathcal{H}\}$ is irreducible, due to proposition 11, it follows that $U(T)U(S)U(TS)^{-1}$ is trivial, i.e. $U(T)U(S)U(TS)^{-1} = c(T,S) \cdot I$, where $c(T,S) \in \mathbf{C}$. Hence

$$U(T)U(S) = c(T,S) \cdot U(TS)$$

with $c(T,S) \in \mathbf{C}$. But the unitarity of $U(\cdot)$ forces $c(T,S)$ to be of modulo one, i.e. $|c(T,S)| = 1$. This means that the mapping $S \to U(S)$ defines a projective representation of the restricted symplectic group $\mathcal{S}p_2(\mathcal{H})$. The group cocycle $c(T,S)$ depends on the choice of the arbitrary phase in $U(S)$. In [Lu 2], Lundberg gave an explicit formula for the cocycle $c(T,S)$, by choosing $U(S)$ such that $c(T,S)$ is smooth in such a way that $U(\cdot)$ lift one-parameter groups into one-parameter groups, for S and T close to the identity, completely analogous to what we did in the orthogonal case (see section 1.4). We do this below, by first giving a constructive proof of the if-part of theorem 15, in the case of S in a neighbourhood of the identity in $\mathcal{S}p_2(\mathcal{H})$ of the form $S = e^A$, with $A \in sp_2(\mathcal{H})$. This is done by first constructing the metaplectic representation on a Lie algebra level, that is we construct $U(e^{sA})$, for $A \in sp_2(\mathcal{H})$ and $s \in \mathbf{R}$, sufficiently small, by first constructing its skew-selfadjoint generator $dU(A)$, hence $U(e^{sA})$ is given by $e^{s \cdot dU(A)}$.

Consider first the complex linear part A_1 of $A \in sp_2(\mathcal{H})$, which is skew-selfadjoint. In this case $dU(A_1)$ and $U(e^{sA_1})$ are constructed by the method of Cook's second quantization given in section 2.2 (where we denoted $U(e^{sA_1})$ by $U(sA_1)$). Hence $dU(A_1)$ is skew-selfadjoint on $\mathcal{D}$ and $\mathcal{D}$ form a dense set of analytic vectors for $dU(A_1)$, since A and then A_1 are bounded. Observe that $dU(A_1) : \vee^n\mathcal{H} \to \vee^n\mathcal{H}$.

We now turn to the anti-linear part A_2 of $A \in sp_2(\mathcal{H})$. So A_2 is Hilbert-Schmidt and selfadjoint (in contrast to the orthogonal case). Due to the Hilbert-Schmidt condition of A_2, there exist two orthogonal sets $\{u_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ in $\mathcal{H}$, both spanning the range of A_2, such that A_2 has the representation

$$A_2 f = \sum_{i\in I} \langle f, v_i\rangle\, u_i$$

for any $f \in \mathcal{H}$. A direct calculation, using the definition of the adjoint of an anti-linear operator and the selfadjointness of A_2, gives that

$$\sum_{i\in I} \langle f, v_i\rangle\, u_i = \sum_{i\in I} \langle f, u_i\rangle\, v_i$$

Notice that this equation differs from the analougous equation in the orthogonal case, by a minus sign. So we identify A_2 with a vector $\mathcal{A}_2 \in \vee^2\mathcal{H}$, where

$$\mathcal{A}_2 = \sum_{i\in I} v_i \vee u_i = \sqrt{2} \cdot \sum_{i\in I} v_i \otimes u_i$$

Observe that $\left\|\frac{1}{\sqrt{2}}\mathcal{A}_2\right\|^2 = \|\sum_{i\in I} v_i \otimes u_i\| = \sum_{i\in I} \|v_i\|^2 \cdot \|u_i\|^2$.

We have chosen $\mathcal{A}_2$ such that the mapping $A_2 \mapsto \frac{1}{\sqrt{2}}\mathcal{A}_2$ becomes an isometry with respect to the Hilbert-Schmidt norm on A_2. In fact, $\|A_2\|_{HS}^2 = \sum_{j\in\mathbf{N}} \|A_2 e_j\|^2$, where we may choose the orthonormal basis such that $e_i = \frac{v_i}{\|v_i\|}$ for $i \in I$, then

$$\|A_2\|_{HS}^2 = \sum_{j\in\mathbf{N}} \left\| \sum_{i\in I} \langle e_j, v_i\rangle \, u_i \right\|^2 = \sum_{j\in\mathbf{N}} \sum_{i\in I} \delta_{i-j} \|v_i\|^2 \cdot \|u_i\|^2 = \sum_{i\in I} \|v_i\|^2 \cdot \|u_i\|^2$$

Notice that the self-adjointness of A_2 mentioned above, is the reason why the above construction gives us a vector $\mathcal{A}_2 \in \vee^2\mathcal{H}$. In the orthogonal case A_2 was skew-selfadjoint reflecting in the sign difference mentioned above, and the analougous construction gave a vector in $\wedge^2\mathcal{H}$.

We are now able to map A_2 into an operator on $\mathcal{F}_\vee(\mathcal{H})$ by generalizing the idea of the creation operator, as we did in the orthogonale case. Define the operator $a^*(A_2)$ on product vectors by

$$a^*(A_2)\Omega = \mathcal{A}_2 \tag{2.19}$$

and

$$a^*(A_2)(f_1 \vee \cdots \vee f_n) = \mathcal{A}_2 \vee f_1 \vee \cdots \vee f_n \tag{2.20}$$

Notice that the vector $\mathcal{A}_2 \vee f_1 \wedge \cdots \wedge f_n$ is well-defined in $\vee^{n+2}\mathcal{H}$ whenever $\mathcal{A}_2 \in \vee^2\mathcal{H}$ takes the form of a finite linear combination of simple product vector, $\mathcal{A}_2 = \sum_{i=1}^N g_{1,i} \vee g_{2,i} \in \vee^2\mathcal{H}$, where $N \in \mathbf{N}$. Any vector $\mathcal{A}_2 \in \vee^2\mathcal{H}$ can be approximate by finite linear combination of such product vectors. Let $\mathcal{A}_2$ be an arbitrary vector in $\vee^2\mathcal{H}$ and $\mathcal{A}_2^{(N)} = \sum_{i=1}^N g_{1,i} \vee g_{2,i}$ an approximating sequence, i.e. $\mathcal{A}_2 = \lim_{N\to\infty} \mathcal{A}_2^{(N)}$, in $\vee^2\mathcal{H}$. Then we may define $\mathcal{A}_2 \vee f_1 \wedge \cdots \wedge f_n$ as the limit of $\mathcal{A}_2^{(N)} \vee f_1 \wedge \cdots \wedge f_n$, in $\vee^{n+2}\mathcal{H}$, as N tends to infinity. The limit depends only of the vector $\mathcal{A}_2$ (and ofcause of $f_1 \wedge \cdots \wedge f_n$), but not on the choise of approximating sequence $\mathcal{A}_2^{(N)}$, since $\|\mathcal{A}_2 \vee f_1 \wedge \cdots \wedge f_n\| \leq \sqrt{(n+2)(n+1)} \cdot \|\mathcal{A}_2\|$ as follows from below. Moreover it also follows that $a^*(A_2)$ is an unbounded operator, with $\mathcal{D}$ a dense set of analytic vectors for $a^*(A_2)$. However, we will first prove that $a^*(A_2)$ as an operator from $\vee^n\mathcal{H}$ into $\vee^{n+2}\mathcal{H}$ is bounded, for each $n \in \mathbf{N} \cup \{0\}$. Even that the resulting bound is the same as in the anti-symmetric case, the calculations become more complicated in this case due to the actual symmetry. Notice that $\|a^*(A_2)\Omega\| = \|\mathcal{A}_2\| = \|\mathcal{A}_2\| \cdot \|\Omega\|$, so we may extend $a^*(A_2)$ by linearity and continuity to a bounded operator from $\vee^0\mathcal{H}$ into $\vee^2\mathcal{H}$ and bound $\|a^*(A_2)|_{\vee^0\mathcal{H}}\| = \|\mathcal{A}_2\|$. Now, let $n \in \mathbf{N}$ and consider $a^*(A_2)$ from $\vee^n\mathcal{H}$ into $\vee^{n+2}\mathcal{H}$. Since $\{v_i\}_{i\in I}$ is an orthogonal set, which spans the range of A_2 in $\mathcal{H}$, we may choose an orthonormal basis $\{e_j\}_{j\in\mathbf{N}}$ for $\mathcal{H}$ such that $e_i = \frac{v_i}{\|v_i\|}$, for $i \in I$. Then

$$\mathcal{A}_2 = \sum_{i\in I} \|v_i\|\, e_i \vee \sum_{j\in I} \langle e_j, u_i\rangle\, e_j = \sum_{i\in I}\sum_{j\in I} \alpha_{ij} e_i \vee e_j$$

where $\alpha_{ij} = \langle e_j, u_i\rangle \cdot \|v_i\|$ and we may choose $\alpha_{ij} = \alpha_{ji}$, due to the symmetry. So $\|\mathcal{A}_2\|^2 = 2 \cdot \sum_{i,j\in I} |\alpha_{ij}|^2$.

Now any normalized n-particle basis vector $E^{(r)_m}_{(k)_m} \in \vee^n\mathcal{H}$, with $\sum_{l=1}^m r_l = n$, may be written as

$$E^{(r)_m}_{(k)_m} = K_{(r)_m} \cdot e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m}$$

where we have used our convention of multiindex and our choice of orthonormal basis (and normalization) given in the end of section 1.1, especially is $K_{(r)_m} = \prod_{l=1}^m (r_l!)^{-\frac{1}{2}}$. Then

$$\begin{aligned}
&\left\| a^*(A_2) E^{(r)_m}_{(k)_m} \right\|^2 \\
&= \left\| \sum_{i\in I}\sum_{j\in I} K_{(r)_m} \cdot \alpha_{ij} e_i \vee e_j \vee e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m} \right\|^2 \\
&= \sum_{i,j\in I}\sum_{s,t\in I} \overline{\alpha_{ij}} \alpha_{st} K^2_{(r)_m} \left\langle e_i \vee e_j \vee e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m}, e_s \vee e_t \vee e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m} \right\rangle
\end{aligned}$$

By use of the inner product formula derived in section 1.1 we get

$$\left\langle e_i \vee e_j \vee e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m}, e_s \vee e_t \vee e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m} \right\rangle = K^{-2}_{(r)_m} \cdot c^{s,t}_{i,j}$$

where $c^{s,t}_{i,j} = 0$, for $\{i,j\} \neq \{s,t\}$, and for $s = i$ and $t = j$ we get

$$c^{i,j}_{i,j} = \begin{cases} 1 & \text{for } i,j \notin J \text{ and } i \neq j \\ 2 & \text{for } i = j \notin J \\ r_p + 1 & \text{for } i \notin J \text{ and } j = k_p \in J \\ r_q + 1 & \text{for } i = k_q \in J \text{ and } j \notin J \\ (r_p+1)(r_q+1) & \text{for } i = k_q \in J, j = k_p \in J \text{ and } i \neq j \\ (r_p+2)(r_p+1) & \text{for } i = j = k_p \in J \end{cases}$$

where $J = \{k_1, \ldots, k_m\}$, hence for all $i, j \in I$ we have

$$\begin{aligned} c_{i,j}^{i,j} &\leq \left(\sum_{p=1}^{m} k_p + 2\right)\left(\sum_{q=1}^{m} k_q + 1\right) \\ &= (n+2)(n+1) \end{aligned}$$

In case $s = j$ and $t = i$ we get, completely analogously, that

$$c_{i,j}^{j,i} \leq (n+2)(n+1)$$

by a symmetry consideration. Then

$$\begin{aligned} \left\|a^*(A_2)E_{(k)_m}^{(r)_m}\right\|^2 &= \sum_{i,j\in I}\left(\alpha_{ij}\cdot\overline{\alpha_{ij}}\cdot c_{i,j}^{i,j} + \alpha_{ij}\cdot\overline{\alpha_{ji}}\cdot c_{i,j}^{j,i}\right) \\ &= \sum_{i,j\in I}|\alpha_{ij}|^2\cdot c_{i,j} \end{aligned}$$

since $\alpha_{ij} = \alpha_{ji}$ and where $c_{i,j} = c_{i,j}^{i,j} + c_{i,j}^{j,i}$.

Hence

$$\begin{aligned} \left\|a^*(A_2)E_{(k)_m}^{(r)_m}\right\|^2 &\leq 2\cdot(n+2)(n+1)\cdot\sum_{i,j\in I}|\alpha_{ij}|^2 \\ &= (n+2)(n+1)\cdot\|\mathcal{A}_2\|^2 \\ &= (n+2)(n+1)\cdot\|\mathcal{A}_2\|^2\cdot\left\|E_{(k)_m}^{(r)_m}\right\|^2 \end{aligned}$$

We notice that this upper bound is actually reached in certain cases, for example for $I = \{i_0\}$ and $e_{k_1}^{r_1} \vee \cdots \vee e_{k_m}^{r_m} = e_{i_0}^n$ is $E_{(k)_m}^{(r)_m} = (n!)^{-\frac{1}{2}}\cdot e_{i_0}^n$, $\mathcal{A}_2 = \|\mathcal{A}_2\|\cdot e_{i_0}^2$ and $\left\|a^*(A_2)E_{(k)_m}^{(r)_m}\right\|^2 = \|\mathcal{A}_2\|^2\cdot(n!)^{-1}\cdot\left\|e_{i_0}^{n+2}\right\|^2 = \|\mathcal{A}_2\|^2\,\frac{(n+2)!}{n!} = (n+2)(n+1)\,\|\mathcal{A}_2\|^2\cdot\left\|E_{(k)_m}^{(r)_m}\right\|^2$. Consequently we may extend $a^*(A_2)$, as an operator mapping $\vee^n\mathcal{H} \to \vee^{n+2}\mathcal{H}$, by linearity and continuity getting a bound operator denoted by the same symbol,

$$\|a^*(A_2)F_n\| \leq \sqrt{(n+2)(n+1)}\cdot\|\mathcal{A}_2\|\cdot\|F_n\|$$

It is remarkable that we get the same bound as we did in the orthogonal case (of course the operators act in different spaces), remembering that the creation and annihilation operators were bounded in contrast to the actual case. Of cause $a^*(A_2)$, as an operator on $\mathcal{F}_\vee(\mathcal{H})$, is unbounded (in the above example is $\left\|a^*(A_2)(n!)^{-\frac{1}{2}}e_{i_0}^n\right\| = \sqrt{(n+2)(n+1)}\cdot\|\mathcal{A}_2\|$, which tends to infinity as n do). Now it follows that $a^*(A_2)$ defines a densely defined unbounded operator on $\mathcal{F}_\vee(\mathcal{H})$, with $\mathcal{D}$ as a dense set of analytic vectors, by completely similar reasons as those used in the anti-symmetric case (see section 1.4). Moreover, $a^*(A_2) : \vee^n\mathcal{H} \to \vee^{n+2}\mathcal{H}$.

Consider $\langle a^*(f_1)F_n, \mathcal{A}_2\rangle$, for arbitrary $f_1 \in \mathcal{H}$ and for any product vector $F_n \in \vee^n\mathcal{H}$, $n \in \mathbf{N}\cup\{0\}$. It is evidently zero for $n \neq 1$. For $n = 1$ we put $F = f_2$, whereby

$$\begin{aligned}
\langle a^*(f_1)f_2, \mathcal{A}_2\rangle &= \langle f_1 \vee f_2, \mathcal{A}_2\rangle \\
&= \sum_{i\in I}\left(\langle f_1, v_i\rangle\langle f_2, u_i\rangle + \langle f_1, u_i\rangle\langle f_2, v_i\rangle\right) \\
&= \sum_{i\in I}\langle f_2, \langle f_1, v_i\rangle u_i + \langle f_1, u_i\rangle v_i\rangle \\
&= \langle f_2, 2A_2 f_1\rangle
\end{aligned}$$

Since $\langle F, 2A_2f_1\rangle = 0$, for $F \notin \vee^1\mathcal{H}$, we have

$$\langle a^*(f_1)F, \mathcal{A}_2\rangle = \langle F, 2A_2f_1\rangle$$

for all product vectors F, hence for all $F \in \mathcal{D}$. Then $\mathcal{A}_2 \in \mathcal{D}(a(f))$ and $a(f)\mathcal{A}_2 = 2A_2f$. Then

$$\begin{aligned}
a(f)a^*(A_2)F_n &= a(f)(\mathcal{A}_2 \vee F_n) \\
&= (a(f)\mathcal{A}_2) \vee F_n + \mathcal{A}_2 \vee (a(f)F_n) \\
&= \left(2a^*(A_2f) + a^*(A_2)a(f)\right)F_n
\end{aligned}$$

for any $F_n = f_1 \vee \cdots \vee f_n \in \vee^n\mathcal{H}$, where we have used that $\mathcal{D}$ is invariant under both $a(f)$ and $a^*(A_2)$. Hence

$$\frac{1}{2}\cdot[a(f), a^*(A_2)] = a^*(A_2f) \tag{2.21}$$

on $\mathcal{D}$, for all $f \in \mathcal{H}$.

We denote the formal adjoint $a^*(A_2)^*$ of $a^*(A_2)$ by $a(A_2)$. In a similar manner as in the symmetric case it follows that $a(A_2)$ is well-defined on $\mathcal{D}$

and that it is inductively given on productvectors $f_1 \vee \cdots \vee f_n \in \mathcal{D}(a(A_2))$ by

$$a(A_2)(f_1 \vee \cdots \vee f_n) = (a^*(f_1)a(A_2) + 2a(A_2 f_1))(f_2 \vee \cdots \vee f_n)$$

and $a(A_2)\Omega = a(A_2)f = 0$, for $f \in \mathcal{D}(a(A_2))$. Observe that $a(A_2) : \vee^n\mathcal{H} \to \vee^{n-2}\mathcal{H}$, yielding zero for $n = 0, 1$.

Moreover, it follows that $\mathcal{D}$ is a dense set of analytic vectors for $a(A_2)$, the proof of the anti-symmetric case carries over to the present case without trouble.

Since $\mathcal{D}$ is invariant under the action of $a^*(f)$ the adjoint of formula (2.21) also holds on $\mathcal{D}$, i.e. $\frac{1}{2}[a(A_2), a^*(f)] = a(A_2 f)$, for all $f \in \mathcal{H}$. Thus $\mathcal{D}$ is invariant under the action of both $a^*(A_2)$ and $a(A_2)$.

Define $dU(A_2) = \frac{1}{2}(a(A_2) - a^*(A_2))$ on $\mathcal{D}$, for any selfadjoint anti-linear Hilbert-Schmidt operator A_2. Notice that $dU(A_2)$ is skew-symmetric by definition and that $dU(A_2) : \vee^n\mathcal{H} \to \vee^{n-2}\mathcal{H} \oplus \vee^{n+2}\mathcal{H}$. Define $dU(A) = dU(A_1) + dU(A_2)$ on $\mathcal{D}$, for any $A = A_1 + A_2 \in sp_2(\mathcal{H})$. Observe that $dU(A) : \vee^n\mathcal{H} \to \vee^{n-2}\mathcal{H} \oplus \vee^n\mathcal{H} \oplus \vee^{n+2}\mathcal{H}$

Theorem 17 *The operator $dU(A)$ is essentially skew-selfadjoint and fulfils*

$$[dU(A), \pi(f)] = \pi(Af) \tag{2.22}$$

on $\mathcal{D}$ for $f \in \mathcal{H}$ and all $A \in sp_2(\mathcal{H})$,

$$\langle \Omega, dU(A)\Omega \rangle = 0 \tag{2.23}$$

for all $A \in sp_2(\mathcal{H})$, and

$$\langle \Omega, dU(A)dU(B)\Omega \rangle = -\frac{1}{4}\langle \mathcal{A}_2, \mathcal{B}_2 \rangle = -\frac{1}{2} \cdot \mathrm{Tr}(B_2 A_2) \tag{2.24}$$

Proof. Since $dU(A)$ is skew-symmetric and has $\mathcal{D}$ as a dense set of analytic vectors, it follows that $dU(A)$ is essentially skew-selfadjoint, by a modification of Nelson's theorem ([R-S 2, p. 202], modified to skew-symmetric operators). From theorem 15 we have that $[dU(A_1), \pi(f)] = \pi(A_1 f)$ on $\mathcal{D}$, for all $f \in \mathcal{H}$ (since it holds for π replaced by a and a^* respectively).

For the anti-linear part of A we get, quite analogue to the proof of (1.9) in theoren 4, that

$$\begin{aligned}[dU(A_2), \pi(f)] &= \frac{1}{2}[a(A_2), \pi(f)] - \frac{1}{2}[a^*(A_2), \pi(f)] \\ &= \pi(A_2 f)\end{aligned}$$

and

$$\begin{aligned}[dU(A),\pi(f)] &= [dU(A_1),\pi(f)]+[dU(A_2),\pi(f)]\\ &= \pi(A_1f)+\pi(A_2f)=\pi(Af)\end{aligned}$$

on $\mathcal{D}$, for all $f\in\mathcal{H}$, proving (2.22). Moreover, since $dU(A_1)\Omega=0$ and $a(A_2)\Omega=0$, it follows that $\langle\Omega,dU(A)\Omega\rangle=-\frac{1}{2}\langle\Omega,a^*(A_2)\Omega\rangle=0$, proving (2.23). Finally

$$\begin{aligned}\langle\Omega,dU(A)dU(B)\Omega\rangle &= -\langle dU(A)\Omega,dU(B)\Omega\rangle\\ &= -\frac{1}{4}\langle a^*(A_2)\Omega,a^*(B_2)\Omega\rangle\\ &= -\frac{1}{4}\langle\mathcal{A}_2,\mathcal{B}_2\rangle\\ &= -\frac{1}{2}\mathrm{Tr}(B_2,A_2)\end{aligned}$$

on $\mathcal{D}$, where we have used the anti-symmetry of $dU(A)$ and (2.19). The spectral forms of A_2 and B_2, respectively, give the last equality as follows. Put

$$A_2f=\sum_{i\in I}\langle f,v_i\rangle u_i=\sum_{i\in I}\langle f,u_i\rangle v_i$$

and

$$B_2f=\sum_{j\in J}\langle f,x_j\rangle y_j=\sum_{j\in J}\langle f,y_j\rangle x_j$$

such that $\mathcal{A}_2=\sum_{i\in I}v_i\vee u_i$ and $\mathcal{B}_2=\sum_{j\in J}x_j\vee y_j$. Choose an orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ for $\mathcal{H}$ such that $e_i=\frac{v_i}{\|v_i\|}$, for $i\in I$. Then

$$\begin{aligned}\langle\mathcal{A}_2,\mathcal{B}_2\rangle &= \sum_{i\in I}\sum_{j\in J}\langle v_i\vee u_i,x_j\vee y_j\rangle\\ &= \sum_{i\in I}\sum_{j\in J}(\langle v_i,x_j\rangle\langle u_i,y_j\rangle+\langle v_i,y_j\rangle\langle u_i,x_j\rangle)\\ &= \sum_{i\in I}\left(\left\langle u_i,\sum_{j\in J}\langle v_i,x_j\rangle y_j\right\rangle+\left\langle u_i,\sum_{j\in J}\langle v_i,y_j\rangle x_j\right\rangle\right)\\ &= 2\cdot\sum_{i\in I}\langle u_i,B_2v_i\rangle\end{aligned}$$

By $u_i = A_2 \frac{v_i}{\|v_i\|^2}$ and $A_2 e_k = 0$ for $k \notin I$, we get

$$\begin{aligned}
\langle \mathcal{A}_2, \mathcal{B}_2 \rangle &= 2 \cdot \sum_{i \in I} \left\langle A_2 \frac{v_i}{\|v_i\|}, B_2 \frac{v_i}{\|v_i\|} \right\rangle \\
&= 2 \cdot \sum_{k \in \mathbf{N}} \langle A_2 e_k, B_2 e_k \rangle \\
&= 2 \cdot \sum_{k \in \mathbf{N}} \langle e_k, B_2 A_2 e_k \rangle \\
&= 2 \cdot \mathrm{Tr}(B_2 A_2)
\end{aligned}$$

where we also have used the definition of the adjoint of an anti-linear operator B_2 together with the self-adjointness of B_2 proving equation (2.24). ∎

We now define the unitary one-parameter group $U(e^{sA})$ by

$$U\left(e^{sA}\right) = e^{s \cdot dU(A)} \tag{2.25}$$

At this point we are rather close to have proved the if part of theorem 16 for $S = e^{sA}$, $A \in sp_2(\mathcal{H})$, in a neighbourhood of the identity. However, the fact that $U(e^{sA})$ create an infinite number of particles and that $\pi(f)$ is unbounded complicate the last part of the proof. Now, since

$$\begin{aligned}
\frac{1}{t}\left(U(e^{tA}) - I\right) U\left(e^{tA}\right) F &= U\left(e^{sA}\right) \frac{1}{t}\left(U\left(e^{sA}\right) - I\right) F \\
&\to U\left(e^{sA}\right) dU(A) F
\end{aligned}$$

for $F \in \mathcal{D}$, as $t \to 0$, it follows that the left hand side also converges, so $U(e^{sA})F \in \mathcal{D}(dU(A))$ and $dU(A)U(e^{sA})F = U(e^{sA})dU(A)F$ (remembering that $dU(A)$ also denotes the skew-selfadjoint closure of the corresponding (preclosed) essentially skew-selfadjoint operator, denoted by the same symbol).

We will show that $U(e^{sA})\pi(f) = \pi(e^{sA}f)U(e^{-sA})$, however, the right hand side is not immediately well-defined. But

$$\pi(g) \sum_{n=0}^{m} \frac{s^n}{n!} dU(-A)^n F$$

is well-defined for $F \in \mathcal{D}$ and $g \in \mathcal{H}$ arbitrary, since $dU(-A)\mathcal{D} \subset \mathcal{D}$. The series $\sum_{n=0}^{m} \frac{s^n}{n!}\pi(g)dU(-A)^n F$ converges in norm. In fact

$$\|\pi(g)dU(-A)^n F\| \leq \sqrt{2\cdot(N+2n+1)}\cdot\|g\|\cdot\|dU(-A)^n F\|$$

by formula (2.10), where N have been chosen such that $F_k = 0$, for $k > N$ and we have used that $dU(-A)^n F \in \oplus_{l=0}^{N+2n} \vee^l \mathcal{H}$. Moreover, $\sum_{n=0}^{m} \frac{s^n}{n!}\|dU(-A)^n F\|$ converges for s sufficiently small, since $F \in \mathcal{D}$ is an analytic vector for $dU(-A)$, so

$$\frac{s}{n+1}\cdot\frac{\|dU(-A)^{n+1}F\|}{\|dU(-A)^n F\|} \to 0 \quad \text{as } n \to \infty$$

for $|s|$ less that some $s_0 \in \mathbf{R}_+$ (or $dU(-A)^n F = 0$ for n larger that some $N \in \mathbf{N}$). Then also

$$\begin{aligned}
&\frac{s}{n+1}\cdot\sqrt{\frac{2(N+2n+3)}{2(N+2n+1)}}\cdot\frac{\|dU(-A)^{n+1}F\|}{\|dU(-A)^n F\|} \\
&= \frac{s}{n+1}\cdot\sqrt{1+\frac{2}{N+2n+1}}\cdot\frac{\|dU(-A)^{n+1}F\|}{\|dU(-A)^n F\|}
\end{aligned}$$

tends to zero as n tends to infinity for $|s|$ less that $s_0 \in \mathbf{R}_+$, giving the norm convergence of the series $\pi(g)\sum_{n=0}^{m}\frac{s^n}{n!}dU(-A)^n F$ for arbitrary $g \in \mathcal{H}$, especially for $g = e^{-sA}f$. Hence by the closedness of $\pi(g)$

$$\lim_{m\to\infty}\sum_{n=0}^{m}\pi(g)\frac{s^n}{n!}dU(-A)^n F = \pi(g)\lim_{m\to\infty}\sum_{n=0}^{m}\frac{s^n}{n!}dU(-A)^n F = \pi(g)U(e^{sA})$$

proving that $\pi(g)U(e^{sA})$ is well-defined on $\mathcal{D}$ for all $g \in \mathcal{H}$, especially for g of the form $g = e^{sA}f$, where $f \in \mathcal{H}$. (Notice that $\pi(g)$ denote the selfadjoint closure of the essentially selfadjoint operator $\frac{1}{\sqrt{2}}(a(g) + a^*(g))$).

To continue, we will show the formula

$$dU(-A)^n\pi(f) = \sum_{k=0}^{n}\binom{n}{k}\pi((-A)^k f)dU(-A)^{n-k}$$

on $\mathcal{D}$, for all $n \in \mathbf{N}$ and $k = \{0,\ldots,n\}$, by induction. For $n = 0$ $(k = 0)$, the formula is trivially true. For $n = 1$ $(k = 0, 1)$ we get by (2.22) that $dU(-A)\pi(f) = \pi(f)dU(-A) + \pi((-A)f)$, which clearly equals the right hand side, as it should. Assume that we have shown the formula for a given $n \in \mathbf{N}$. Then

$$\begin{aligned} dU(-A)^{n+1}\pi(f) &= dU(-A)\sum_{k=0}^{n}\binom{n}{k}\pi((-A)^k f)dU(-A)^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}\pi((-A)^k f)dU(-A)^{n+1-k} \\ &\quad + \sum_{k=0}^{n}\binom{n}{k}\pi((-A)^{k+1} f)dU(-A)^{n-k} \end{aligned}$$

by the assumption and formula (2.22). By separating out the first term ($k = 0$) in the first sum, the last term ($k = n$) in the second sum and moreover make the transformation $k \to k-1$ in the rest of the second sum, we get

$$\begin{aligned} & dU(-A)^{n+1}\pi(f) \\ = & \binom{n}{0}\pi((-A)^0 f)dU(-A)^{n+1} + \binom{n}{n}\pi((-A)^{n+1} f)dU(-A)^0 \\ & + \sum_{k=1}^{n}\binom{n}{k}\pi((-A)^k f)dU(-A)^{n+1-k} \\ & + \sum_{k=1}^{n}\binom{n}{k-1}\pi((-A)^k f)dU(-A)^{n+1-k} \end{aligned}$$

By the Pascal triangle (or direct calculation) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ and the fact that $\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$ we finally get the desired formula for $n+1$,

$$dU(-A)^{n+1}\pi(f) = \sum_{k=0}^{n+1}\binom{n+1}{k}\pi((-A)^k f)dU(-A)^{n+1-k}$$

and the induction is completed.

Using the just derived formula we have

$$\begin{aligned} & U\left(e^{sA}\right)\pi(f)F \\ = & \lim_{m\to\infty}\sum_{n=0}^{m}\frac{s^n}{n!}dU(-A)^n\pi(f)F \end{aligned}$$

$$= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{s^n}{n!} \sum_{k=0}^{m} \binom{n}{k} \pi((-A)^k f) dU(-A)^{n-k} F$$
$$= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{s^n}{n!} \frac{d^n}{ds^n}\bigg|_{s=0} \pi\left(e^{-sA} f\right) U\left(e^{-sA}\right) F$$

where we have used that $\pi(f)F \in \mathcal{D}$, for any $F \in \mathcal{D}$, and that $\pi(e^{-sA}f)U(e^{-sA})F$ is well-defined and C^∞ with respect to $s \in \mathbf{R}$ for any $F \in \mathcal{D}$, in fact

$$\begin{aligned}
&\frac{d^n}{ds^n} \pi\left(e^{-sA} f\right) U\left(e^{-sA}\right) F \\
&= \sum_{k=0}^{n} \binom{n}{k} \pi\left((-A)^k e^{-sA} f\right) dU(-A)^{n-k} U\left(e^{-sA}\right) F \\
&= \sum_{k=0}^{n} \binom{n}{k} \pi\left(g_k\right) U\left(e^{-sA}\right) \left(dU(-A)^{n-k} F\right)
\end{aligned}$$

is clearly well-defined inductively by the argument above and where $g_k = (-A)^k e^{-sA} f$, since $dU(-A)^{n-k}F \in \mathcal{D}$ for all $k \in \{0, \ldots, n\}$. Again by the same argument we get

$$\begin{aligned}
U\left(e^{-sA}\right) \pi(f) F &= \lim_{m\to\infty} \sum_{n=0}^{m} \frac{s^n}{n!} \frac{d^n}{ds^n}\bigg|_{s=0} \pi\left(e^{-sA} f\right) U\left(e^{-sA}\right) F \\
&= \pi\left(e^{-sA} f\right) U\left(e^{-sA}\right) F
\end{aligned}$$

proving that $U(e^{-sA})\pi(f) = \pi(e^{-sA}f)U(e^{-sA})$ on $\mathcal{D}$, for all $f \in \mathcal{H}$, all $A \in sp_2(\mathcal{H})$ and s in a sufficiently small neighbourhood of zero.

We thereby get the formula

$$U(S)^{-1} \pi(f) U(S) = \pi(S^{-1} f) = \pi_S(f) \tag{2.26}$$

on $\mathcal{D}$, for all $S = e^{sA}$, $A \in sp_2(\mathcal{H})$, $f \in \mathcal{H}$ and $s \in \mathbf{R}$ in a neighbourhood of zero, where $U(S)$ has been explicitely constructed, such that the arbitrary phase of $U(S)$ has been fixed on all one-parameter subgroups of $\mathcal{S}p_2(\mathcal{H})$ of the form $S = e^{sA}$, $A \in sp_2(\mathcal{H})$.

We call $U : S \to U(S)$ *the metaplectic representation* of the restricted symplectic group. It turns out that the elements $dU(A)$, $A \in sp_2(\mathcal{H})$, form a Lie algebra $mp_2(\mathcal{H})$, called *the metaplectic Lie algebra* corresponding to *the metaplectic group* $\mathcal{M}p_2(\mathcal{H})$ defined as the group of all unitary implementers $U(S)$, $S \in \mathcal{S}p_2(\mathcal{H})$ from Shale's theorem (theorem 16).

Theorem 18 *The bracket in the metaplectic Lie algebra is given by*

$$[dU(A), dU(B)] = dU([A, B]) + \omega(A, B) \cdot I \qquad (2.27)$$

on $\mathcal{D}$, for any $A, B \in sp_2(\mathcal{H})$, with the Lie algebra cocycle given by

$$\begin{aligned} \omega(A, B) &= \frac{1}{2}\mathrm{Tr}([A_2, B_2]) \\ &= -\frac{i}{2} Im \langle \mathcal{A}_2, \mathcal{B}_2 \rangle \end{aligned} \qquad (2.28)$$

So $A \to dU(A)$ defines a projective representation of the Lie algebra $sp_2(\mathcal{H})$ of the restricted symplectic group onto the metaplectic Lie algebra $mp_2(\mathcal{H})$, in $\mathcal{F}_\vee(\mathcal{H})$.

Proof. From (2.26) and (2.11) it follows that

$$U(S^{-1})W(f)U(S) = W(S^{-1}f)$$

for all $f \in \mathcal{H}$ and $S \in \mathcal{S}p_2$. Then

$$\begin{aligned} & U(e^{tA})U(e^{sB})U(e^{-tA})W(f)U(e^{tA})U(e^{sB})U(e^{-tA}) \\ &= W\left(e^{tA}e^{sB}e^{-tA}f\right) \\ &= W\left(e^{s \cdot C(t)}\right) \\ &= U\left(e^{s \cdot C(t)}\right) W(f) U\left(e^{-s \cdot C(t)}\right) \end{aligned}$$

for all $f \in \mathcal{H}$, where $A, B, C(t) \in \mathcal{S}p_2(\mathcal{H})$ and $C(t)$ is given from A and B by the formula $C(t) = e^{tA}Be^{-tA}$, derived in the proof of theorem 2 part 3), of chapter 1. Thus the irreducibility of the CCR-algebra shown in corollary 12, gives

$$U(e^{tA})U(e^{sB})U(e^{-tA}) = b(tA, sB) \cdot U(e^{-s \cdot C(t)})$$

Of course this result also follows by consideration of the irreducibility of $*$-algebra generated by $\{\pi(f) : f \in \mathcal{H}\}$ instead, using proposition 11. The unitarity of $U(\cdot)$ implies that $|b(tA, sB)| = 1$. Moreover, from the defining equation it follows directly that $b(0, sB) = b(tA, 0) = 1$. Now

$$b(tA, (s+r)B)\ U(e^{-(s+r)C(t)}) = b(tA, sB)b(tA, rB)\ U(e^{-(s+r)C(t)})$$

by the above and the one-parameter group properties. That is $s \to b(tA, sB)$ is a homomorphism. Furthermore $t \to b(tA, sB)$ is analytic. Then it follows that we may write

$$b(tA, sB) = e^{c(tA,sB)}$$

where $c(tA, 0) = c(0, sB) = 0$, $t \to c(tA, sB)$ is analytic and $c(tA, sB)$ is linear in s, so $c(tA, sB) = s \cdot c(tA, B)$, i.e.

$$b(tA, sB) = e^{s \cdot c(tA,B)}$$

Then

$$\begin{aligned}
&U(e^{tA})dU(B)U(e^{tA}) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(U(e^{tA})U(e^{sB})U(e^{-tA})\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(e^{s \cdot c(tA,B)}U(e^{-s \cdot C(t)})\right) \\
&= c(tA, B) \cdot I + dU(C(t))
\end{aligned}$$

on $\mathcal{D}$, and

$$\begin{aligned}
&U(e^{tA})[dU(A), dU(B)]U(e^{-tA}) \\
&= \frac{d}{dt}\left(U(e^{tA})dU(B)U(e^{-tA})\right) \\
&= \frac{d}{dt}(c(tA, B) \cdot I + dU(C(t))) \\
&= c'(tA, B) \cdot I + dU(e^{tA}[A, B]e^{-tA})
\end{aligned}$$

on $\mathcal{D}$, where we have used that $\frac{d}{dt}C(t) = e^{tA}[A, B]e^{-tA}$ and $c'(tA, B) = \frac{d}{dt}c(tA, B)$. Hence, for $t = 0$, $[dU(A), dU(B)] = dU([A, B]) + \omega(A, B) \cdot I$ on $\mathcal{D}$, where $\omega(A, B) = \frac{d}{dt}\left.\right|_{t=0} c(tA, B)$.

Taking the vacuum expectation value of this equation we get

$$\begin{aligned}
\omega(A, B) &= \langle \Omega, ([dU(A), dU(B)] - dU([A, B]))\Omega \rangle \\
&= \langle \Omega, dU(A)dU(B)\Omega \rangle - \langle \Omega, dU(B)dU(A)\Omega \rangle \\
&= -\frac{1}{2}\mathrm{Tr}(B_2 A_2) + \frac{1}{2}\mathrm{Tr}(A_2 B_2) \\
&= \frac{1}{2}\mathrm{Tr}([A_2, B_2])
\end{aligned}$$

where we have used (2.23) and (2.24). Since $\frac{1}{2}\mathrm{Tr}(A_2B_2) - \frac{1}{2}\mathrm{Tr}(B_2A_2) = \frac{1}{4}\langle \mathcal{B}_2, \mathcal{A}_2\rangle - \frac{1}{4}\langle \mathcal{A}_2, \mathcal{B}_2\rangle = -\frac{i}{2}Im\,\langle \mathcal{A}_2, \mathcal{B}_2\rangle$, all claims of the theorem is proved. ∎

We remark that the commutator of two anti-linear Hilbert-Scmidt operators does not generally vanish. Moreover, it follows directly from (2.27) that ω is skew-symmetric and fulfils the Hochschild condition

$$\omega(AB, C) + \omega(BC, A) + \omega(CA, B) = 0$$

for all $A, B, C \in sp_2(\mathcal{H})$. Then the Jacobi identity is also fulfilled

$$\omega([A, B], C) + \omega([B, C], A) = \omega([C, A], B) = 0$$

for all $A, B, C \in sp_2(\mathcal{H})$, by which ω is a closed two-form. In the special case, when the linear part of A and B are trace-class operators, we are able to transform the cocycle term away, by a change of phase, as follows. Put $dU_0(A) = dU(A) + \frac{1}{2}\mathrm{Tr}(A_1)\cdot I$. Then a straight forward calculation gives

$$\begin{aligned} [dU_0(A), dU_0(B)] &= [dU(A), dU(B)] \\ &= dU([A, B]) + \omega(A, B)\cdot I \\ &= dU_0([A, B]) \end{aligned}$$

since $\omega(A, B) = \frac{1}{2}\mathrm{Tr}([A, B]_1)$ by

$$\begin{aligned} \mathrm{Tr}([A, B]_1) &= \mathrm{Tr}([A_1, B_1] + [A_2, B_2]) \\ &= 0 + \mathrm{Tr}([A_2, B_2]) \\ &= 2\cdot\omega(A, B) \end{aligned}$$

So we put $U_0(e^{sA}) = e^{s\cdot dU_0(A)} = e^{\frac{1}{2}s\cdot\mathrm{Tr}(A_1)}U(e^{sA})$, for $s \in \mathbf{R}$, close to zero. Then

$$U(e^{sC}) = e^{\frac{1}{2}s(\mathrm{Tr}(C_1) - \mathrm{Tr}(A_1) - \mathrm{Tr}(B_1))}U(e^{sA})U(e^{sB})$$

where C is given by the Campbell-Baker-Hausdorff formula, such that $e^{sC} = e^{sA}e^{sB}$ for s close to zero, i.e. the group cocycle $c(e^{sA}, e^{sB})$ is given by

$$c(e^{sA}, e^{sB}) = (\det(e^{sA_1}e^{sB_1}e^{-sC_1}))^{\frac{1}{2}}$$

for s close to zero, where $\det(e^D) = e^{\mathrm{Tr}(D)}$ for any trace class operator D, by definition.

We conclude this section by calculating an explicit formula for *the vacuum functional*, given by $c(s) = \left\langle \Omega, U(e^{sA})\Omega \right\rangle$, for $A \in sp_2(\mathcal{H})$ and s in a neighbourhood of zero. Excactly as in the anti-symmetric case we arrive with

$$\Omega_s = c(s) \cdot e^{-\frac{1}{2}a^*(K)}\Omega$$

where $K = S_2 S_1^{-1}$ and $S = e^{sA}$ for s sufficiently small. This formula allows us to get a differential equation for $c(s)$, as follows

$$\begin{aligned}
c'(s) &= \frac{d}{ds} \langle \Omega, \Omega_s \rangle = \langle \Omega, dU(A)\Omega_s \rangle \\
&= \langle -dU(A)\Omega, \Omega_s \rangle = \frac{1}{2} \langle a^*(A_2)\Omega, \Omega_s \rangle \\
&= \frac{1}{2} c(s) \left\langle \mathcal{A}_2, e^{-\frac{1}{2}a^*(K)}\Omega \right\rangle = -\frac{1}{4} c(s) \langle \mathcal{A}_2, \mathcal{K} \rangle \\
&= -\frac{1}{2} \cdot \mathrm{Tr}(K A_2) \cdot c(s)
\end{aligned}$$

where we have used formula (2.25), the formula for Ω_s derived above, and the fact that $dU(A)\Omega = -\frac{1}{2}a^*(A_2)\Omega$. So $c(s)$ is given by the differential equation above and the fact that $c(0) = \|\Omega\|^2 = 1$. Notice that we have the opposite sign than in the corresponding anti-symmetric case. Remark that $K = S_2 S_1^{-1} = (e^{sA})_2 (e^{sA})_1^{-1}$ depends on $s \in \mathbf{R}$. Put $V_s = e^{-sA_1}(e^{sA})_1$ then

$$\frac{d}{ds} \mathrm{Tr}(\log V_s) = \overline{\mathrm{Tr}(K A_2)}$$

Completely as in the anti-symmetric case. We may now rewrite the differential equation as

$$c'(s) = -\frac{1}{2} \frac{d}{ds} \overline{\mathrm{Tr}(\log V_s)} \cdot c(s)$$

which has the solution

$$c(s) = \kappa \cdot e^{-\frac{1}{2}\overline{\mathrm{Tr}(\log V_s)}} = \kappa \cdot \left(\overline{\det(V_s)}\right)^{-\frac{1}{2}} = \kappa \cdot (\det(V_{-s}))^{-\frac{1}{2}}$$

since $c(-s) = \overline{c(s)}$. In the previous paragraphs we have used that the determinant of V_s does exist, since $V_s - I = \int_0^s e^{-tA_1} A_2 (e^{tA})_2 \, dt$ is a trace-class operator, because both A_2 and $(e^{tA})_2$ are Hilbert-Schmidt operators (see for example [R-S 3, p. 322] or [Ar, p. 124]). Finally it follows that $\kappa = 1$, since $c(0) = 1$, whereby

$$c(s) = (\det(V_{-s}))^{-\frac{1}{2}}$$

giving an explicit expression for the vacuum functional, as claimed. Notice that this formula describes another power-law than in the similar expression for the spin representation. As a historical remark, we notice that $c(s) \neq 0$, so Shale's choice of fixing the phase by demanding $c(s) > 0$ was indeed possible (see [Sh, p. 157, theorem 4.1]). However, Shales choice is different from ours which fixes the phase by demanding that $U(S)$ should be a one-parameter group if S is one. We summarise the above as follows.

Theorem 19 *The vacuum functional $c(s) = \langle\Omega, U(e^{sA})\Omega\rangle$ for $A \in sp_2(\mathcal{H})$ and s in a neighbourhood of zero, where $U(\cdot)$ denote the metaplectic representation, is simply*

$$c(s) = (\det(V_{-s}))^{-\frac{1}{2}}$$

where $V_{-s} = e^{sA_1}(e^{-sA})_1 = I - \int_0^s e^{tA_1} A_2 (e^{-tA})_2 \, dt$.

Proof. A imediately consequence of the above. ∎

The metaplectic construction, derived in this chapter, will be used in chapter 4 of applications, to construct some special representations of the diffeomorphism group $\mathit{Diff}^+(S^1)$ and of the Virasoro algebra in the symmetric Fock Hilbert space. But before doing so, we shall consider the loop algebras and the Virasoro algebra in general. Moreover, we'll construct som special representations of the Virasoro in the anti-symmetric Fock Hilbert space.

Chapter 3

Loop Algebras and the Virasoro Algebra

3.1 Loop algebras.

In this section we introduce some basic facts on loop algebras and related subjects. The reason for doing so is partly to give a useful frame for the Virasoro algebra, which is considered in the next section, and partly to obtain knowledge on the loop group, which will be studied in a later chapter. We notice that the diffeomorphism group $Diff^{+}(S^1)$ act as a group of automorphisms on loop groups and algebras and that the central extension of the Lie algebra $Vect(S^1)$ of $Diff^{+}(S^1)$ indeed is the Virasoro algebra. Moreover the central extendsion of the loop algebra, together with the further extension, discussed below, are nice example of Kac-Moody algebras, in fact they are socalled affine Kac-Moody algebras (see [K-R,p.93-98], [P-S,p.76-78] and [Mi,p.21-23]). Futhermore, the simplest representation of the Kac-Moody algebras is given in terms of the spin representation, for details we refer to [Ar,p.124] (see also [Ve,p.1]). Hereby the connection between the spin representation and the loop algebra is seen. There is an analogue connection between the metaplectic representation and the Virasoro algebra, however the analogue construction is more cumbersome [Ve,p.1]. In chapter 4 of applications we will show, in details, how these subjects are related for some particular cases.

Since 1984 a huge number of papers have appeared on these subjects. The reason for this increasing interest is related to the important and succesful use in two dimensional conformal theories and string theories in physics, see [L-T], [Mi], [M-S] and [C-T], see also the sub-introduction in section 3.2.

We will base this section on a few of all the appearing papers, namely [P-S], [K-R], [Hu], [Mi], [G-W 1] and [G-W 2], as follows. The introduction of the loop algebras and the affine Kac-Moody algebras given below is a compound of the considerations given in [K-R, chapter 9], [G-W 2, paragraph 1], [G-W 2, section 1] together with [Mi, chapter 2]. The discussion concerning the Casimir operator and its generalization is a result of [P-S, section 9.4], [G-W 1, paragraph 2] and [K-R, chapter 10]. Moreover, we will sometimes refer to results from [Hu].

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbf{C}$, i.e. it is a non-Abelian finite dimensional Lie algebra with no non-trivial ideals. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot,\cdot]_0$. Let $\mathbf{C}[t,t^{-1}]$ denote the ring of Laurent polynomials in $t \in \mathbf{C}$ and $t^{-1} \in \mathbf{C}$ and put

$$\tilde{\mathfrak{g}}_t = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$$

If $\mathbf{B} \subset \mathfrak{g}$ denotes a basis for $\mathfrak{g}$, then we denote the element $x \otimes t^n$ by $x(n)$, where $x \in \mathbf{B}$ and $t^n \in \mathbf{C}[t, t^{-1}]$, $n \in \mathbf{Z}$, is in the canonical basis for $\mathbf{C}[t, t^{-1}]$. Then

$$\{x(n) = x \otimes t^n : x \in \mathbf{B}, n \in \mathbf{Z}\}$$

form a basis for $\tilde{\mathfrak{g}}_t$. Notice the abuse of notation, since $x(n)$ is not the image of a point n under a mapping x, however, the used notation is quite common in this context.

We are mainly interested in the restriction from $t \in \mathbf{C} \setminus \{0\}$ to $t \in S^1 = \{z \in \mathbf{C} : |z| = 1\}$. In this case we sometimes write $t = e^{i\theta}$, where $\theta \in [0, 2\pi]$, as we write $\tilde{\mathfrak{g}}$ (without the index t) for the span of $\{x(n) = x \otimes e_n : x \in \mathbf{B}, n \in \mathbf{Z}\}$ where $e_n = e^{in\theta}$.

Sometimes one benefits from viewing $\tilde{\mathfrak{g}}$ as the space of smooth mappings $S^1\mathfrak{g}$, which is called loops, from S^1 to the Lie algebra $\mathfrak{g}$, i.e. we identify $\tilde{\mathfrak{g}}$ with $S^1\mathfrak{g}$. Notice that $S^1\mathfrak{g}$ is naturally an infinite-dimensional Lie algebra through the pointwise commutator

$$[x(m), y(n)]_\sim = [x, y]_0(m+n)$$

Here $S^1\mathfrak{g}$ is simply a vector space by pointwise addition of loops and natural multiplication of loops with scalars.

The above identification is in fact responsible for the name *loop algebra* for $\tilde{\mathfrak{g}} = S^1\mathfrak{g}$. Furthermore we may identify $\tilde{f} \in \tilde{\mathfrak{g}}$ with $\sum_{n \in \mathbf{Z}} x_n e^{in\theta}$ where x_n in $\mathfrak{g}$ tends to zero as $|n|$ approach infinity, hence we may write $\tilde{f}$ in $\tilde{\mathfrak{g}}$ as a Fourier series with coefficients x_n in $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a group G, then $\tilde{\mathfrak{g}} = S^1\mathfrak{g}$ is the Lie algebra of the *loop group LG*, consisting of all smooth mappings from S^1 into G (see for example [P-S, p. 27]). As above we may identify $\tilde{\mathfrak{g}}_t$ with the space of all formal Laurent mappings from $\mathbf{C} \setminus \{0\}$ to the Lie algebra $\mathfrak{g}$ and we may write an element $\tilde{f} \in \tilde{\mathfrak{g}}_t$ as

$$\tilde{f}(t) = \sum_{n \in \mathbf{Z}} x_n t^n$$

with $x_n \in \mathfrak{g}$.

Now let $(\cdot,\cdot)_0$ denote the Killing form on $\mathfrak{g}$, i.e. $(\cdot,\cdot)_0$ is the bilinear symmetric invariant form on $\mathfrak{g}$ given by

$$(x,y)_0 = \mathrm{Tr}\,(\mathrm{ad}(x)\circ\mathrm{ad}(y))$$

here $\mathrm{ad}(x)$ is given by $(\mathrm{ad}(x))(z)=[x,z]$, in the adjoint representation, for further details see for example [Mi, p. 4] or [Hu, p. 21].

Here invariance means that

$$([x,y]_0,z)_0 = (x,[y,z]_0)_0$$

for all $x,y,z\in\mathfrak{g}$.

We may then define a skew-symmetric bilinear form ω on $\tilde{\mathfrak{g}}_t$ by

$$\omega(f,g) = \mathrm{res}_0\left(\frac{df}{dt},g\right)_0$$

for $f,g\in\tilde{\mathfrak{g}}_t$, where res_0 denote the residue at zero. Then for arbitrary $x,y\in\mathfrak{g}$ we have

$$\begin{aligned}\omega\left(x(m),y(n)\right) &= m\cdot(x,y)_0\cdot\mathrm{res}_0 t^{m+n-1}\\ &= m\cdot(x,y)_0\cdot\delta_{m+n}\end{aligned}$$

observe that ω is in fact skew symmetric and bilinear on $\tilde{\mathfrak{g}}_t$. Alternatively for arbitrary $f,g\in\tilde{\mathfrak{g}}_t$ we may use the integral formula to calculate the residue as

$$\omega(f,g) = \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{df\left(e^{i\theta}\right)}{d\theta},g\left(e^{i\theta}\right)\right)_0 d\theta$$

From the invariance and the symmetry of the Killing form $(\cdot,\cdot)_0$ on $\mathfrak{g}$, it follows that ω defines a two-cocycle, by use of the above formula

$$\begin{aligned}\omega\ &\left(x(m),[y(n),z(k)]_\sim\right)+\omega\left(y(n),[z(k),x(m)]_\sim\right)+\omega\left(z(k),[x(m),y(n)]_\sim\right)\\ =\ &\left(m\cdot(x,[y,z]_0)_0+n\cdot(y,[z,x]_0)_0+k\cdot(z,[x,y]_0)_0\right)\cdot\delta_{m+n+k}\\ =\ &0\end{aligned}$$

proving our claim. There is essentially no other cocycle on $\tilde{\mathfrak{g}}_t$ than ω [P-S, p. 40]. The above construction also holds for the restriction of t to S^1. Notice that the two-cocycle ω is invariant under the action of the group $Diff^+(S^1)$ of orientation preserving diffeomorphisms of the circle, as

$$\omega(f_\phi, g_\phi) = \omega(f, g)$$

where $f_\phi = f \circ \phi$ and $g_\phi = g \circ \phi$ for $\phi \in Diff^+(S^1)$. We return to a discussion of $Diff^+(S^1)$ later on. However this means that $Diff^+(S^1)$ acts as a group of automorphisms of the extended Lie algebra, which we study below, since

$$[\phi(x(m)), \phi(y(n))]_\sim = \phi([x(m), y(n)]_\sim)$$

by direct calculations, it follows that ϕ is a isomorphism and then an automorphism of $S^1\mathfrak{g}$ because $\phi(x(m)) \in S^1\mathfrak{g}$.

Define $\hat{\mathfrak{g}}'_t = \tilde{\mathfrak{g}}_t \oplus \mathbf{C} \cdot c$ and $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbf{C} \cdot c$ as the *central extensions* of $\tilde{\mathfrak{g}}_t$ and $\tilde{\mathfrak{g}}$ respectively. Both are Lie algebras with respect to the bracket or commutator given by

$$\begin{aligned} [f, g] &= [f, g]_\sim + \omega(f, g) \cdot c \\ [f, c] &= 0 \end{aligned}$$

for $f, g \in \tilde{\mathfrak{g}}_t$ respectively $f, g \in \tilde{\mathfrak{g}}$. The element c is of course central. Expressed in terms of basis vectors this gives

$$[x(m), y(n)] = [x, y]_0(m+n) + m \cdot \delta_{m+n} \cdot (x, y)_0 \cdot c$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbf{Z}$. Here $[x, y]_0(m+n)$ denotes $[x, y]_0 \otimes t^{m+n}$ by our notational convention. The Lie algebras $\hat{\mathfrak{g}}'_t$ and $\hat{\mathfrak{g}}'$ are called *affine Kac-Moody algebras* associated with $\mathfrak{g}$. Note that $\hat{\mathfrak{g}}'_t$ is generated by $\{x(-1), x(0), x(1) : x \in \mathfrak{g}\}$ and that the mapping $x \mapsto x(0)$ is a Lie algebra isomorphism from $\mathfrak{g}$ into $\hat{\mathfrak{g}}'_t$ i.e. it is invertible, linear and conserves the commutator operation. Naturally the same holds for $\hat{\mathfrak{g}}'$ instead of $\hat{\mathfrak{g}}'_t$, since the first is the restriction of the second to $t \in S^1$.

We will identify $\mathfrak{g}$ with its image in $\hat{\mathfrak{g}}'_t$. We can now extend the Killing form from $\mathfrak{g}$ to $\hat{\mathfrak{g}}'_t$ by defining

$$\begin{aligned} (f, g) &= \operatorname{res}_0\left(t^{-1}(f(t), g(t))_0\right) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(f\left(e^{i\theta}\right), g\left(e^{i\theta}\right)\right)_0 d\theta \end{aligned}$$

and

$$(f,c) = (c,c) = 0$$

for $f,g \in \tilde{\mathfrak{g}}_t$. Notice that the form is indeed symmetric, invariant and bilinear, but degenerated due to $c \in \hat{\mathfrak{g}}'_t$. On basis vectors $x(m), y(n) \in \tilde{\mathfrak{g}}_t$ we get $(x(m), y(n)) = (x,y)_0 \cdot \delta_{m+n}$. Moreover we may decompose $\hat{\mathfrak{g}}'_t$ as

$$\hat{\mathfrak{g}}'_t = \oplus_{n \in \mathbf{Z}} \mathfrak{g}(n) \oplus \mathbf{C} \cdot c$$

where $\mathfrak{g}(n)$ is spanned by $\{x(n) : x \in \mathfrak{g}\}$, identifying $\mathfrak{g}(0)$ with $\mathfrak{g}$, whereby $\hat{\mathfrak{g}}'_t$ is generated by $\mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$. Of course, we get the same picture if we use $\hat{\mathfrak{g}}'$ instead of $\hat{\mathfrak{g}}'_t$.

As mentioned above $\hat{\mathfrak{g}}'_t$ and $\hat{\mathfrak{g}}'$ are both degenerated with respect to the extended Killing form, due to the central element c. To avoid this inconvenience we will extend $\hat{\mathfrak{g}}'_t$ and $\hat{\mathfrak{g}}'$ once more. This is done in the following paragraphs.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ (for further details see for example [Hu, p. 80] or [Mi, p. 5]), where Λ is the corresponding root system, Λ^+ is a system of positive roots, $\Lambda^- = -\Lambda^+$ implying $\Lambda = \Lambda^+ \cup \Lambda^-$ and $\Lambda_s \subset \Lambda^+$ is a systems of simple roots. The elements of Λ^+ can be written as a sum of elements from Λ_s. Now let $\mathfrak{g}_\alpha$ denote the (one-dimensional) root space corresponding to the root $\alpha \in \Lambda$. Choose $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and $y_\alpha \in \mathfrak{g}_{-\alpha} \setminus \{0\}$ for every $\alpha \in \Lambda^+$. The commutator on $\hat{\mathfrak{g}}'_t$ respectively $\hat{\mathfrak{g}}'$ therefore gives

$$\begin{aligned}
[h, x_\alpha(n)] &= [h, x_\alpha]_0(0+n) + n \cdot \delta_n \cdot (h, x_\alpha)_0 \cdot c \\
&= [h, x_\alpha]_0(n) \\
&= \alpha(h) x_\alpha(n)
\end{aligned}$$

and similar

$$\begin{aligned}
[h, y_\alpha(n)] &= [h, y_\alpha]_0(n) \\
&= -\alpha(h) y_\alpha(n)
\end{aligned}$$

We also have that

$$[c, x_\alpha(n)] = [c, y_\alpha(n)] = [c, h] = 0$$

for $h \in \mathfrak{h}$. If we define a Cartan subalgebra of respectively $\hat{\mathfrak{g}}'_t$ and $\hat{\mathfrak{g}}'$ as $\mathfrak{h} \oplus \mathbf{C} \cdot c$, then each root α has infinite multiplicity. To avoid this, we extend $\hat{\mathfrak{g}}'_t$ respectively $\hat{\mathfrak{g}}'$ by joining an element d, to obtain

$$\hat{\mathfrak{g}}_t = \hat{\mathfrak{g}}'_t \oplus \mathbf{C} \cdot d = \tilde{\mathfrak{g}}_t \oplus \mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$$

and the analogue without the t-index

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbf{C} \cdot d = \tilde{\mathfrak{g}} \oplus \mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$$

Both extensions become Lie algebras with respect to the commutator given by the old one on $\hat{\mathfrak{g}}'_t$ respectively $\hat{\mathfrak{g}}'$ and in addition

$$[d, x(n)] = n \cdot x(n)$$

and

$$[d, c] = 0$$

It follows that

$$[d, f(t)] = t \frac{df(t)}{dt}$$

for any $f \in \hat{\mathfrak{g}}'_t$ respectively $f \in \hat{\mathfrak{g}}'$, since this formula holds on basis vectors of the form $x(n)$. In the case of $\hat{\mathfrak{g}}$, the new element d has the following tangible realization, $d = -i\frac{d}{d\theta}$. This leads us to a definition of the Cartan algebra for $\hat{\mathfrak{g}}_t$ respectively $\hat{\mathfrak{g}}$, as

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$$

We may write the roots in component form as $(\alpha, 0, n)$, corresponding to the above decomposition of $\hat{\mathfrak{h}}$. For example is $(\alpha, 0, n)$ the root of the root vector $x_\alpha(n)$. So the above extension has reduced the root multiplicity to at most one, for $\alpha \in \Lambda \setminus \{0\}$.

For $\alpha = 0$, the roots $(0, 0, n)$ where $n \in \mathbf{Z} \setminus \{0\}$ have multiplicity $\dim(\mathfrak{h})$, since the vectors in $\mathfrak{h}(n)$ span the root subspace corresponding to the root $(0, 0, n)$. We choose to define the set of positive roots as

$$\hat{\Lambda}^+ = \{(\alpha, 0, n) : \alpha \in \Lambda, n \in \mathbf{N}\} \cup \left\{(\alpha, 0, 0) : \alpha \in \Lambda^+\right\}$$

and the set of negative roots as

$$\hat{\Lambda}^- = -\hat{\Lambda}^+$$

so that $\hat{\Lambda} = \hat{\Lambda}^+ \cup \hat{\Lambda}^-$ as in the case of finite dimensional (semi-) simple Lie algebras.

Proposition 20 *The affine Lie algebras $\hat{\mathfrak{g}}_t = \hat{\mathfrak{g}}'_t \oplus \mathbf{C} \cdot d$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbf{C} \cdot d$ carries both a non-degenerated, symmetric, invariant bilinear form $(\cdot, \cdot)$ given by*

$$\begin{aligned}
(f(t), g(t)) &= \operatorname{res}_0 \left(t^{-1} (f(t), g(t))_0\right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(f(e^{i\theta}), g(e^{i\theta})\right)_0 d\theta, \\
(f(t), c) &= (f(t), d) = (d, f(t)) \\
&= (c, c) = 0, \\
(c, d) &= 1
\end{aligned}$$

and

$$(d, d) = s$$

for $f, g \in \tilde{\mathfrak{g}}_t$ respectively $f, g \in \tilde{\mathfrak{g}}$ and $s \in \mathbf{C}$.

Proof. This proposition is a generalization of [K-R, proposition 9.1], where $s = 0$. We extend their proof to the case of $s \neq 0$. In this proof we only discuss the case of $\hat{\mathfrak{g}}_t$ since the case of $\hat{\mathfrak{g}}$ follows in a completely analogous manner. Notice that the form reduce to the old one when restricted to $\hat{\mathfrak{g}}'_t$. Since the old form was symmetric and bilinear the same properties evidently hold for the new form, however, the degeneracy of the old form is apparently eliminated by the demand $(c, d) = 1$. Consequently the extended form is non-degenerated so the only thing left to prove is the invariance of the form. We know that the restriction to $\hat{\mathfrak{g}}'_t$ is invariant, and since $[c, d] = 0$ we just have to prove that

$$([x(m), d], y(n)) = (x(m), [d, y(n)])$$

This follows from a direct calculation:

$$\begin{aligned}
([x(m), d], y(n)) &= (-m \cdot x(m), y(n)) \\
&= -m \cdot (x, y)_0 \cdot \delta_{m+n}
\end{aligned}$$

and

$$\begin{aligned}
(x(m), [d, y(n)]) &= (x(m), n \cdot y(n)) \\
&= n \cdot (x, y)_0 \cdot \delta_{m+n}
\end{aligned}$$

from which the claim follows. ∎

Corollary 21 *If $(\cdot,\cdot)$ is an invariant form on $\hat{\mathfrak{g}}_t$, then it is a positive multiple of the form defined in proposition* 20.

Proof. This corollary and a sketch of the proof can be found in [Mi, proposition 2.2.3]. We present the proof in details. The invariance of the form yields

$$\begin{aligned} n\cdot(x_\alpha(n),x_\beta(m)) &= (([d,x_\alpha(n)],x_\beta(m)) \\ &= -((x_\alpha(n),[d,x_\beta(m)]) \\ &= -m\cdot(x_\alpha(n),x_\beta(m)) \end{aligned}$$

so

$$(n+m)\cdot(x_\alpha(n),x_\beta(m))=0$$

i.e. $(x_\alpha(n),x_\beta(m)) = q_{\alpha\beta}\cdot\delta_{m+n}$, where $q_{\alpha\beta}=q_{\alpha\beta}(n)$ depends on n. If we choose an orthonormal basis with respect to $(\cdot,\cdot)_0$ instead, then the structure constants, given by $[x_\alpha,x_\beta]_0=\sum_\mu a^\mu_{\alpha\beta}x_\mu$, obey that

$$\begin{aligned} a^\gamma_{\alpha\beta} &= \sum_\mu a^\mu_{\alpha\beta}(x_\mu,x_\gamma)_0 \\ &= \sum_\mu a^\mu_{\alpha\gamma}(x_\alpha,x_\mu)_0 = a^\alpha_{\beta\gamma} \end{aligned}$$

Hence it follows that

$$a^\gamma_{\alpha\beta}=a^\alpha_{\beta\gamma}=a^\beta_{\gamma\alpha}$$

Moreover the anti-symmetry of the commutator gives $a^\gamma_{\alpha\beta}=-a^\gamma_{\beta\alpha}$, whereby we have the identities

$$a^\gamma_{\alpha\beta}=a^\alpha_{\beta\gamma}=a^\beta_{\gamma\alpha}=-a^\gamma_{\beta\alpha}=-a^\beta_{\alpha\gamma}=-a^\alpha_{\gamma\beta}$$

We shall especially use that $a^\gamma_{\alpha\beta}=-a^\gamma_{\beta\alpha}$ and that $a^\gamma_{\alpha\beta}=-a^\gamma_{\beta\alpha}$. The invariance of the extended form yields

$$\begin{aligned} \sum_\mu q_{\alpha\mu}\cdot a^\mu_{\gamma\beta} &= \sum_\mu a^\mu_{\gamma\beta}(x_\alpha(n),x_\mu(-n)) \\ &= \sum_\mu a^\mu_{\alpha\gamma}(x_\mu(n),x_\beta(-n)) = \sum_\mu a^\mu_{\alpha\gamma}\cdot q_{\mu\beta} \end{aligned}$$

Then

$$\sum_{\mu} q_{\alpha\mu} \cdot a^{\beta}_{\gamma\mu} = \sum_{\mu} a^{\mu}_{\gamma\alpha} \cdot q_{\mu\beta}$$

since $a^{\mu}_{\gamma\beta} = -a^{\beta}_{\gamma\mu}$ and $a^{\mu}_{\alpha\gamma} = -a^{\mu}_{\gamma\alpha}$.

If we put $A_\gamma = \left\{a^{\mu}_{\gamma\alpha}\right\}_{\mu\alpha}$ and $Q = \{q_{\mu\beta}\}_{\mu\beta}$ then the above equation reads $QA_\gamma = A_\gamma Q$. Now, the adjoint representation with respect to the canonical basis $\{x_\alpha\}$ is represented by A_γ, since

$$(\mathrm{ad}\,(x_\gamma))\,(x_\beta) = [x_\gamma, x_\beta]_0 = \sum_{\mu} a^{\mu}_{\gamma\beta} x_\mu$$

Since the adjoint representation is irreducible for a simple Lie algebra and Q commute with all A_γ it follows, by Schur's lemma (see for example [Hu, p. 26]) that Q is a multiple of the unit matrix I, i.e. $Q = \{q_{\alpha\beta}(n)\}_{\alpha\beta} = q(n) \cdot I$ giving

$$(x_\alpha(n), x_\beta(m)) = q(n) \cdot \delta_{n+m}\delta_{\alpha-\beta}$$

Further use of the invariance of the form gives

$$\begin{aligned} a^{\gamma}_{\alpha\beta} \cdot q(n+1) &= \sum_{\mu} a^{\mu}_{\alpha\beta}\,(x_\mu(n+1), x_\gamma(-(n+1))) \\ &= a^{\beta}_{\gamma\alpha} \cdot q(n) = a^{\gamma}_{\alpha\beta} \cdot q(n) \end{aligned}$$

where we repeatedly have used

$$([x_\alpha(n), x_\beta(m)]\,, x_\gamma(k)) = \left([x_\alpha, x_\beta]_0\,(m+n), x_\gamma(k)\right)$$

Consequently $q(n+1) = q(n) = q$. Hence $(x_\alpha(n), x_\beta(m)) = q \cdot \delta_{n+m} \cdot \delta_{\alpha-\beta}$ and a renormalization of the form, by a constant factor gives $(x_\alpha(n), x_\beta(m)) = \delta_{m+n} \cdot \delta_{\alpha-\beta}$, agreeing with the above definition of the extended form in the proposition. Notice that

$$n \cdot (c, x_\alpha(n)) = (c, [d, x_\alpha(n)]) = ([c,d], x_\alpha(n)) = 0$$

and

$$n \cdot (d, x_\alpha(n)) = (d, [d, x_\alpha(n)]) = ([d,d], x_\alpha(n)) = 0$$

according to which $(c, f) = (d, f) = 0$ for any $f \in \tilde{\mathfrak{g}}_t$. ∎

To complete this section on loop algebras we will state some fundamental results, but first we will define the Verma module for loop algebras. In fact what follows holds for arbitrary affine Lie algebras, we denote it $\hat{\mathfrak{g}}$.

We split $\hat{\mathfrak{g}}$ into subalgebras $\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_-$ and $\hat{\mathfrak{h}}$ where $\hat{\mathfrak{n}}_+$ and $\hat{\mathfrak{n}}_-$ are spanned by the positive and negative roots, respectively, i.e.

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$$

and

$$\hat{\mathfrak{n}}_\pm = \mathrm{span}\{x_\alpha(n) : \alpha \in \Lambda^\pm, n \in \mathbf{Z}\}$$

which corresponds to a redefinition of the previous choise of negative and positive roots for the extended affine Kac-Moody algebra discussed earlier in connection with the loop algebras.

Let $\lambda \in \hat{\mathfrak{h}}^*$ be an arbitrary linear form on $\hat{\mathfrak{h}}$. We then define the Verma module as in the finite dimensional case, by

$$V_\lambda = U(\hat{\mathfrak{g}})/I_\lambda$$

where $U(\hat{\mathfrak{g}})$ denote the universal enveloping algebra of $\hat{\mathfrak{g}}$ (see for example [Hu, p. 90]) and I_λ is the left ideal generated by $\hat{\mathfrak{n}}_+$ and the elements $h - \lambda(h)$, where $h \in \hat{\mathfrak{h}}$. Let $\mu \in \hat{\mathfrak{h}}^* \setminus \{0\}$, then it is a weight of V_λ if there is a $v \in V_\lambda$ such that $h \cdot v = \mu(h) \cdot v$, for all $h \in \hat{\mathfrak{h}}$. If μ is a weight we define

$$V_\lambda(\mu) = \left\{v \in V_\lambda : hv = \mu(h)v, \text{ for all } h \in \hat{\mathfrak{h}}\right\}$$

corresponding to the weight $\mu \in \hat{\mathfrak{h}}^* \setminus \{0\}$. Then we have the following theorem.

Theorem 22 *The following holds for the Verma module V_λ of $\hat{\mathfrak{g}}$ (with the above notation):*

1) $\dim V_\lambda(\lambda) = 1$

2) *The set of weights μ of V_λ are of the form $\mu = \lambda - \sum_{i=1}^{l} r_i\alpha_i$ where $r_i \in \mathbf{N} \cup \{0\}$*

3) *V_λ is a direct sum of its weights subspaces $V_\lambda(\mu)$ and any invariant subspace respects this decomposition.*

4) $\dim V_\lambda(\mu) < \infty$

Proof. This theorem is a generalization of that in [Mi, theorem 1.5.4], where the Lie algebra considered is finite dimensional and semisimple, and the proof given there is modified according to the present case. Let $v = 1 + I_\lambda \in V_\lambda$ and let ϕ denote the representation of $\hat{\mathfrak{g}}$ on V_λ given by the natural action, i.e. $\phi(x)(u + I_\lambda) = xu + I_\lambda$, for $x \in \hat{\mathfrak{g}}$ and $u \in U(\hat{\mathfrak{g}})$. Then is $\phi(x_+)v = 0$ in V_λ, $\phi(h)v = \lambda(h)\cdot v$ in V_λ and $\phi(U(\hat{\mathfrak{g}}))v = V_\lambda$. So v is by definition a highest weight vector of weight λ and ϕ is the highest weight representation of $\hat{\mathfrak{g}}$ on V_λ, given by its natural action.

Now let $\hat{\mathfrak{n}}_+$, $\hat{\mathfrak{n}}_-$ and $\hat{\mathfrak{h}}$ be spanned by x_{β_i}, y_{β_i}, where $i \in \mathbf{N}$, and h_i, $i = 1, \ldots, l$ respectively. Notice that the roots β_i are of the form $(\alpha,0,n)$, $\alpha \in \Lambda^+$ and $n \in \mathbf{Z}$. From the Poincaré-Birkhoff-Witt theorem (see [Hu, p.92]) it follows that any ordered monomial in $U(\hat{\mathfrak{g}})$ can be written as a finite product of the form

$$u = \prod_i y_{\beta_i}^{r_i} \prod_j h_j^{s_j} \prod_k x_{\beta_k}^{p_k}$$

If any $p_k \neq 0$ then $uv = 0$, if any $s_j \neq 0$ then $h_j^{s_j} v = \lambda^{s_j} v$. So any element of V_λ can be written as a linear combination of vectors of the form

$$w = \prod_i y_{\beta_i}^{r_i} v$$

i.e. V_λ is generated by linear combinations of elements of the form $U(\hat{\mathfrak{n}}_-)v$. Subsequently a direct calculation gives that

$$h y_\beta^r = y_\beta^r (h - r \cdot \beta(h))$$

whereof

$$\begin{aligned}
hw &= h \prod_i y_{\beta_i}^{r_i} v \\
&= \left(\prod_i y_{\beta_i}^{r_i}\right)\left(h - \sum_i r_i \cdot \beta_i(h)\right) v \\
&= \left(\prod_i y_{\beta_i}^{r_i}\right)\left(\lambda(h) - \sum_i r_i \cdot \beta_i(h)\right) v \\
&= \left(\lambda(h) - \sum_i r_i \cdot \beta_i(h)\right)\left(\prod_i y_{\beta_i}^{r_i} v\right) \\
&= \mu(h) w
\end{aligned}$$

Hence each $w = \prod_i y_{\beta_i}^{r_i} v$ is an eigenvector of each $h \in \hat{\mathfrak{h}}$ with eigenvalue $\mu(h) = \lambda(h) - \sum_i r_i \cdot \beta_i(h)$, where $r_i \in \mathbf{N} \cup \{0\}$, proving part 2).

Notice that $\lambda = \mu$ if and only if all $r_i = 0$, whereby $V_\lambda(\lambda) = \text{span}\{v\}$ and $\dim V_\lambda(\lambda) = 1$, furnishing a proof of part 1).

Moreover each basis vector of the form $w = \prod_i y_\beta^{r_i} v$ in V_λ belongs to precisely one $V_\lambda(\mu)$, giving that $V_\lambda = \oplus_\mu V_\lambda(\mu)$ proving the first part of 3). Noticing that the number of vectors w, such that $\lambda - \sum_i r_i \cdot \beta_i$ equals a given μ, is finite and that these vectors span $V_\lambda(\mu)$, we observe that $\dim V_\lambda(\mu) < \infty$, proving part 4).

We now turn to the last part of 3). Let W be an invariant subspace of V_λ and choose an arbitrary $w \in W$. Then w can be written as a direct sum of finitely many elements $v_i \in V_\lambda(\mu_i) \setminus \{0\}$, with mutually different μ_i's, i.e.

$$w = \sum_{i=1}^{m} v_i \in \oplus_{i=1}^m V_\lambda(\mu_i)$$

Consequently $h^n w = \sum_{i=1}^m \mu_i^n v_i$ for each $n \in \mathbf{N} \cup \{0\}$ whence

$$\begin{pmatrix} w \\ hw \\ h^2 w \\ \vdots \\ h^{m-1} w \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_m \\ \mu_1^2 & \mu_2^2 & \cdots & \mu_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{m-1} & \mu_2^{m-1} & \cdots & \mu_m^{m-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{pmatrix}$$

Since the matrix is a Vandermonde matrix it is invertible, due to $\mu_i \neq \mu_j$, for $i \neq j$, in fact its determinant is given by

$$\det\{\lambda_j^{i-1}\}_{i,j=1,\ldots,m} = \prod_{i<j} (\mu_i - \mu_j) \neq 0$$

The invariance of W under the action of $h \in \hat{\mathfrak{h}}$ and the invertibility of matrix $\{\lambda_j^{i-1}\}_{i,j=1,\ldots,m}$ then implies that each $v_i \in W$. Hence $w \in \oplus_{i=1}^m (W \cap V_\lambda(\mu_i))$, i.e. $W = \oplus_\mu (W \cap V_\lambda(\mu))$, proving the remaining part of 3). ∎

We now turn to the existence and uniqueness of irreducible highest weight representations of $\hat{\mathfrak{g}}$, in Verma modules.

Theorem 23 *The Verma module V_λ contains a unique maximal invariant proper submodule M_λ, moreover $L_\lambda = V_\lambda/M_\lambda$ carries an irreducible highest weight representation of $\hat{\mathfrak{g}}$ with highest weight λ.*

Proof. This proof is identical to [Mi, theorem 2.4.1] and the proof is simply an expansion of that in [Mi]. Let M be any proper invariant subspace of V_λ. By the above theorem we may write $M = \oplus_\mu M(\mu)$, where $M(\mu) = M \cap V_\lambda(\mu)$. Then $M(\lambda) = \{0\}$, since otherwise the highest weight vector $v \in V_\lambda$ would belong to $M(\lambda)$ and $V_\lambda = U(\hat{\mathfrak{g}})v \subset M(\lambda) \subset M$ contradicting the property of M. Now put M_λ equal to the union of all proper invariant subspaces of V_λ.

Then M_λ becomes a maximal invariant subspace, by construction, and M_λ is proper, since it doesn't contain the highest weight vector $v \in V_\lambda$. Moreover M_λ is evidently unique. Therefore $L_\lambda = V_\lambda/M_\lambda$ contains no invariant proper subspaces and the representation of $\hat{\mathfrak{g}}$ on L_λ given by the natural action is obviously irreducible.

The highest weight vector in L_λ is given by $v_\lambda = v + M_\lambda$, since

$$h(v + M_\lambda) = \lambda(v + M_\lambda)$$

for any $h \in \hat{\mathfrak{h}}$,

$$\phi(x_+)(v + M_\lambda) = M_\lambda$$

for any $x_+ \in \hat{\mathfrak{n}}_+$, and

$$U(\hat{\mathfrak{g}})(v + M_\lambda) = V_\lambda + M_\lambda$$

where we have used that $v = 1 + I_\lambda$ is the highest weight vector for V_λ, with highest weight λ. Hence λ is the highest weight and $v_\lambda = v + M_\lambda$ is the highest weight vector for $\hat{\mathfrak{g}}$ on L_λ. ∎

Finally we will define the *Casimir operator* Ω for $\hat{\mathfrak{g}}$ as a generalization and by use of the Casimir operator Ω_0 for the finite dimensional simple Lie algebras. Further generalizations of the Casimir operator Ω provide us with a highest weight representation of the Virasoro algebra through the Sugawara construction (see for example [G-W 1, p. 82]). The Virasoro algebra will be discussed in the next section.

Choose an orthonormal basis $\{x_i\}$ for the finite dimensional simple Lie algebra $\mathfrak{g}$ relative to $(\cdot,\cdot)_0$ and let λ_{ij} denote the structure coefficients of y

and x_i i.e. $[y, x_i]_0 = \sum_j \lambda_{ij} x_j$ and $-\lambda_{ij} = \lambda_{ji}$ (summation over i's and j's without index sets here means summation over the finite basis index set). Let $y \in \mathfrak{g}$ and $m \in \mathbf{Z}$ be arbitrary chosen and consider first $\sum_i x_i(-n)x_i(n)$, giving

$$\begin{aligned}
&[y(m), \sum_i x_i(-n)x_i(n)] \\
&= \sum_i ([y, x_i]_0(m-n)x_i(n) + m \cdot \delta_{m-n}(y, x_i)_0 \cdot c \cdot x_i(n) \\
&\quad + x_i(-n)[y, x_i]_0(m+n) + m \cdot \delta_{m+n}(y, x_i)_0 \cdot c \cdot x_i(-n)) \\
&= \sum_{i,j} \lambda_{ij} (x_j(m-n)x_i(+n) + x_i(-n)x_j(m+n)) \\
&\quad + \sum_i m \cdot \delta_{|m|-|n|} \cdot (y, x_i)_0 \cdot c \cdot x_i(m) \\
&= \sum_{i,j} \lambda_{ij} (x_j(m-n)x_i(+n) - x_j(-n)x_i(m+n)) \\
&\quad + m \cdot y(m) \cdot c \cdot \delta_{|m|-|n|} \\
&= X_{-n+m} - X_{-n} + m \cdot y(m) \cdot \delta_{|m|-|n|} \cdot c
\end{aligned}$$

where $X_n = \sum_{i,j} \lambda_{ij} x_j(n) x_i(m-n)$ and we have used that

$$[y(m), x(n)] = [y, x]_0(m+n) + m \cdot \delta_{m+n} \cdot (y, x)_0 \cdot c$$

as given earlier, and that $y = \sum_i (y, x_i) x_i$. Define Ω_1 as the formal sum

$$\Omega_1 = -\sum_i \sum_{n \in \mathbf{N}} x_i(-n)x_i(n) - \frac{1}{2} \sum_i (x_i(0))^2$$

thus

$$\begin{aligned}
&[y(m), \Omega_1] \\
&= -\sum_{n \in \mathbf{N}} \left[y(m), \sum_i x_i(-n)x_i(n) \right] - \frac{1}{2} \left[y(m), \sum_i x_i(0)x_i(0) \right] \\
&= -\sum_{n \in \mathbf{N}} (X_{-n+m} - X_{-n}) - m \cdot y(m) \cdot c - \frac{1}{2}(X_m - X_0 + 0) \\
&= -(\frac{1}{2}X_0 + X_1 + \cdots + X_{m-1} + \frac{1}{2}X_m) - m \cdot y(m) \cdot c
\end{aligned}$$

The skew-symmetry of λ_{ij} implies that

$$\begin{aligned}
X_k + X_{m-k} &= \sum_{i,j} \lambda_{ij} \left[x_j(k), x_i(m-k)\right] \\
&= \sum_i \left[[y, x_i]_0, x_i\right]_0 (m) \\
&= \left(\left(\sum_i (\mathrm{ad}(x_i))^2 \right) (y) \right) (m) \\
&= (\Omega_0(y))\,(m)
\end{aligned}$$

for any $k \in \{0, 1, \ldots, m\}$, where $\Omega_0 = \sum_i (\mathrm{ad}(x_i))^2$ denotes the Casimir operator in the adjoint representation of the finite dimensional simple Lie algebra $\mathfrak{g}$ (see for example [Hu, p. 27] or [G-W 1, p. 78]).

It is well known that Ω_0 commutes with every element of $\mathfrak{g}$, whence it belongs to the center of $\mathfrak{g}$ and therefore acts as a multiplication by a constant which we denote $2Q$ for later use. The above remark gives in addition that

$$\begin{aligned}
[y(m), \Omega_1] &= -\frac{1}{2} \sum_{k=0}^{m-1} (X_k + X_{m-k}) - m \cdot y(m) \cdot c \\
&= -m \left(\frac{1}{2} \Omega_0(y) \right) (m) - m \cdot y(m) \cdot c \\
&= -m(Q + c) y(m) \\
&= -(Q + c)[d, y(m)]
\end{aligned}$$

since $[d, y(m)] = m\, y(m)$. Hence is

$$[y(m), \Omega_1 - (Q + c)d] = 0$$

for any $y \in \mathfrak{g}$ and any $m \in \mathbf{Z}$ meaning that $\Omega = \Omega_1 - (Q + c)d$ commute with all of $\hat{\mathfrak{g}}$. It is in this sense that the Casimir element Ω generalize Ω_0. Observe that Ω is, in fact, independent of the choice of basis since Ω_0 and $\sum_i x_i(-n) x_i(n)$ are independent of this choice (see for example [P-S, p. 183]).

We will now generalize the Casimir operator above, but first we will have to introduce what Kac (see [K-R, p. 107]) calls *an admissible representation.* We define a representation of $\hat{\mathfrak{g}}$ on a vector space V, or equivalently a $\hat{\mathfrak{g}}$-module V, to be admissible if for all $x \in \mathfrak{g}$ and every vector $v \in V$ there exists a $N \in \mathbf{N}$, where $N = N(v)$, such that $(x(n))(v) = 0$, for $n > N$.

For $x, y \in \mathfrak{g}$ and $k, n \in \mathbf{Z}$ we define the normal ordering $:x(k)y(n):$ of $x(k)$ and $y(n)$ (in agreement with, for example, [Mi, p. 174], [K-R, p. 108] and [G-W 1, p. 79]) by

$$:x(k)y(n):= \begin{cases} x(k)y(n) & \text{for } k < n \\ \frac{1}{2}(x(k)y(n) + y(n)x(k)) & \text{for } k = n \\ y(n)x(k) & \text{for } k > n \end{cases}$$

The following formal infinite sum

$$T(n) = \frac{1}{2}\sum_{k\in\mathbf{Z}}\sum_i :x_i(k)x_i(n-k):$$

$n \in \mathbf{Z}$ reduce to a finite sum on any admissible space V, where $\{x_i\}$ still denotes an orthonormal basis for $\mathfrak{g}$ with respect to $(\cdot,\cdot)_0$, $i = 1, \dots, \dim\mathfrak{g}$. This is so because for any $m < 0$ there is only a finite number of terms $x_i(m)$ to the right due to the normal ordering and given any $v \in V$ there is only a finite number of indices $m > 0$ such that $x_i(m)v \neq 0$, due to the admissibility condition. This also follows by rewriting of $T(n)$, without normal ordering as

$$T(n) = \frac{1}{2}\sum_i \left(\sum_{k<\frac{1}{2}n} x_i(k)x_i(n-k) + \sum_{k\geq\frac{1}{2}n} x_i(n-k)x_i(k) \right)$$

then for $k < n - N(v)$ the first term gives zero acting on $v \in V$ and for $k > N(v)$ the second term gives zero acting on $v \in V$, i.e.

$$\begin{aligned} &(T(n))(v) \\ &= \frac{1}{2}\sum_i \left(\sum_{n-N\leq k<\frac{1}{2}n} x_i(k)x_i(n-k) + \sum_{\frac{1}{2}n\leq k\leq N} x_i(n-k)x_i(k) \right)(v) \\ &= \left(\frac{1}{2}\sum_i \sum_{n-N\leq k\leq N} :x_i(k)x_i(n-k): \right)(v) \end{aligned}$$

Moreover since $[x_i(k), x_i(n-k)] = [x_i, x_i]_0(n) = 0$ for $n \neq 0$ (because then the term $k \cdot \delta_{k+n-k} \cdot (x_i, x_i)_0 \cdot c$ vanish), it follows that we may discard the normal ordering for $n \neq 0$. Moreover, since $k \in \mathbf{Z}$ we may also rewrite $T(n)$ as

$$T(n) = \sum_{k\in\mathbf{Z}} :x_i(-k)x_i(n+k):$$

Observe that $T(n)$ is homogeneous of degree n relative to d, i.e.

$$[d, T(n)] = nT(n)$$

when applied to a $v \in V$. This follows since $[d, x_i(m)] = mx_i(m)$ so

$$dx_i(k)x_i(m) = x_i(k)(d+k)x_i(m) = x_i(k)x_i(m)(d+m+k)$$

hence $[d, x_i(k)x_i(m)] = (k+m)x_i(k)x_i(m)$ and for $k+m=n$, the claim follows. Notice that $T(n)$ is independent of choice of orthonormal basis, since $\sum_i x_i(k)x_i(n-k)$ is.

Since $T(0)$ is closely related to the Casimir operator, in fact $-T(0)$ equals the previous defined operator Ω, we call the operators $T(n)$, $n \in \mathbf{Z}$, for the *shifted Casimir operators*, due to the following lemma:

Lemma 24 *For $y \in \mathfrak{g}$ and $m, n \in \mathbf{Z}$ are*

$$[y(m), T(n)] = m(Q+c)y(m+n)$$

as operators on V (admissible) where $2Q$ denotes the eigenvalue of the ordinary Casimir operator Ω_0 in the adjoint representation of $\mathfrak{g}$.

Proof. This lemma is almost identical to [G-W 1, lemma 2.1] and [K-R, proposition 10.1]. However, their proofs are diferent from our. We use the same idea as we did when we considered the Casimir operator. But first we notice that the commutator of $:x_i(-k)x_i(n+k):$ with any element $y(m)$ is independent of the normal ordering, since it only differs from $x_i(-k)x_i(n+k)$ by a constant multiplied by c, which commute with $y(m)$. Furthermore, we observe that

$$\begin{aligned}
\sum_i [[y, x_i]_0(p), x_i(q)] &= \sum_i [[y, x_i]_0, x_i]_0 (p+q) \\
&= (\Omega_0(y))(p+q) \\
&= 2 \cdot Q \cdot y(p+q)
\end{aligned}$$

for any $p, q \in \mathbf{Z}$, since the term

$$p \cdot \delta_{p+q} \left([y, x_i]_0, x_i\right)_0 \cdot c = p \cdot \delta_{p+q} \left(y, [x_i, x_i]_0\right)_0 \cdot c = 0$$

Due to the admissibility of the representation on V all infinite sums reduce to finite sums, when acting on V. Alternatively one could use a cutoff procedure (see [K-R, p. 16]) and consider

$$T_\varepsilon(n) = \frac{1}{2}\sum_i \sum_{k\in\mathbf{Z}} :x_i(-k)x_i(n+k):\, \chi(\varepsilon\cdot k)$$

which is a finite sum, since $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| > 1$, and subsequently letting ε approach zero. We use the first approach, though.

$$\begin{aligned}
&[y(m), T(n)] \\
&= \left[y(m), \frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}} :x_i(-k)x_i(n+k):\right] \\
&= \frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}} ([y,x_i]_0(m-k)x_i(n+k) + x_i(-k)[y,x_i]_0(m+n+k)) \\
&\quad + \frac{1}{2}\cdot m\cdot c\cdot \sum_{k\in\mathbf{Z}} \left(\delta_{m-k}\cdot \sum_i (y,x_i)_0 x_i(n+k) + \delta_{m+n+k}\cdot \sum_i (y,x_i)_0 x_i(-k)\right) \\
&= \frac{1}{2}\sum_i \left(\sum_{k\ge\frac{m-n}{2}} + \sum_{k<\frac{m-n}{2}}\right)[y,x_i]_0(m-k)x_i(n+k) \\
&\quad + \frac{1}{2}\sum_i \left(\sum_{k\ge-\frac{m+n}{2}} + \sum_{k<-\frac{m+n}{2}}\right) x_i(-k)[y,x_i]_0(m+n+k) \\
&\quad + \frac{1}{2}\cdot m\cdot c\,(y(n+m) + y(m+n))
\end{aligned}$$

where the first sum term has been split into sums which are normal ordered (for $k \ge \frac{m-n}{2}$ respectively $k \ge -\frac{m+n}{2}$) and sums which are not normal ordered (for $k < \frac{m-n}{2}$ respectively $k < -\frac{m+n}{2}$). By use of the above commutator relations we turn the sums of the not normal ordered terms into sums of normal ordered terms. Then

$$\begin{aligned}
[y(m), T(n)] &= \frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}} :[y,x_i]_0(m-k)x_i(k+n): \\
&\quad + \frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}} :x_i(-k)[y,x_i]_0(m+n+k):
\end{aligned}$$

$$+\sum_{k<\frac{m-n}{2}} Q\cdot y(m+n) - \sum_{k<-\frac{m+n}{2}} Q\cdot y(m+n)$$
$$+m\cdot y(n+m)\cdot c$$

Observe that $Q\cdot y(m+n)$ is independent of k, so the sums involving this term simply reduce to $m\cdot Q\cdot y(m+n)$. Hence, by the transformation $k\to k+m$, in the first sum we get

$$[y(m),T(n)] =$$
$$\frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}} :([y,x_i]_0(-k)x_i(k+m+n)+x_i(-k)[y,x_i]_0(k+m+n)):$$
$$+m\cdot(Q+c)\cdot y(n+m)$$

Now the terms in the sum cancel out when summing over i, since for any $y\in\mathfrak{g}$ and $p,q\in\mathbf{Z}$ we have that

$$\begin{aligned}\sum_i x_i(p)[y,x_i]_0(q) &= \sum_{i,j} x_i(p)\left([y,x_i]_0,x_j\right)_0 x_j(q)\\ &= -\sum_{i,j} x_i(p)\left(x_i,[y,x_j]_0\right)_0\cdot x_j(q)\\ &= -\sum_j [y,x_j]_0(p)x_j(q)\\ &= -\sum_i [y,x_i]_0(p)x_i(q)\end{aligned}$$

where we have used the invariance of the Killing form. Finally we obtain the desired formula

$$[y(m),T(n)] = m\cdot(Q+c)\cdot y(n+m)$$

proving the lemma. ∎

Theorem 25 *Under the same hypothesis as in the above lemma, we have that*

$$[T(n),T(m)] = (Q+c)\cdot(n-m)T_{n+m}+\delta_{n+m}\cdot\frac{n^3-n}{12}\cdot\dim\mathfrak{g}\cdot c\cdot(Q+c)$$

Proof. This theorem connects [K-R, theorem 10.1] and [G-W 1, lemma 2.2]. However, our proof utilizes the admissibility of the representation, which seems more appropriate in the present context. As in the above lemma, since we consider an admissible representation, the infinite sums appearing below becomes finite, when applied to any vector. It follows that

$$\begin{aligned}
[T(n),T(m)] &= \frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}}[x_i(-k)x_i(k+n),T(m)] \\
&= \frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}} x_i(-k)(Q+c)(k+n)x_i(k+n+m) \\
&\quad +\frac{1}{2}\sum_i\sum_{k\in\mathbf{Z}}(Q+c)(-k)x_i(-k+m)x_i(k+n)
\end{aligned}$$

Using the commutator relation to normal ordering the terms which are not already normal ordered, i.e. for $k<-\frac{m+n}{2}$ in the first sum and for $k<\frac{m-n}{2}$ in the second sum. Then we get

$$\begin{aligned}
&[T(n),T(m)] \\
&= \frac{1}{2}(Q+c)\sum_i\sum_{k\in\mathbf{Z}}(k+n):x_i(-k)x_i(k+n+m): \\
&\quad +\frac{1}{2}(Q+c)\sum_{k<-\frac{m+n}{2}}(k+n)(-k)\cdot\delta_{-k+k+n+m}\cdot\dim\mathfrak{g}\cdot c \\
&\quad +\frac{1}{2}(Q+c)\sum_i\sum_{k\in\mathbf{Z}}(-k):x_i(-k+m)x_i(k+n): \\
&\quad +\frac{1}{2}(Q+c)\sum_{k<\frac{m-n}{2}}(-k)(-k+m)\cdot\delta_{-k+m+k+n}\cdot\dim\mathfrak{g}\cdot c
\end{aligned}$$

Making the transformation $k\to k+m$ in the third sum gives

$$\begin{aligned}
&[T(n),T(m)] = \\
&\qquad \frac{1}{2}(Q+c)\sum_i\sum_{k\in\mathbf{Z}}(n-m):x_i(-k)x_i(k+n+m): \\
&+\quad \frac{1}{2}(Q+c)\cdot c\cdot\dim\mathfrak{g}\cdot\delta_{n+m}\left(\sum_{k<0}-k(k+n)+\sum_{k<-n}-k(-k-n)\right)
\end{aligned}$$

Hence

$$[T(n), T(m)] = (Q+c)(n-m)T_{n+m} + \delta_{n+m}\frac{n^3-n}{12}\cdot \dim \mathfrak{g}\cdot c\cdot (Q+c)$$

where we have used that the last sums reduce to $\sum_{k=1}^{n-1} k(n-k) = \frac{n^3-n}{6}$ proving the theorem. ∎

In any representation the operator c act as multiplication by a scalar which we also denote c. Let us consider any admissible representation. If $c \neq -Q$ it is customary to consider

$$L_n = \frac{1}{(Q+c)}\cdot T(n)$$

instead of $T(n)$. The L_n, $n \in \mathbf{Z}$ then fulfil the commutation relations

$$[L_n, L_m] = (n-m)\cdot L_{n+m} + \delta_{n+m}\cdot \frac{n^3-n}{12}\cdot \frac{c\cdot \dim \mathfrak{g}}{Q+c}$$

and

$$[L_n, x(m)] = -m\cdot x(m+n)$$

for all $m \in \mathbf{Z}$ and $x \in \mathfrak{g}$. Beware that Q depends on $\mathfrak{g}$ (see [K-R, p. 111]). The construction above is known as *the Sugawara construction* and as we will see in the next section it provides a representation of the socalled Virasoro algebra (since it fulfils the commutation relations of the Virasoro algebra).

The loop algebra and the loop group have been studied intensively in the latest few years, and there is a lot more to be said about these subjects, than we will do here. For further reading we refer to [Jø], [Mi], [P-S], [G-W 1], [G-W 2] and [K-R]. However we will notice that the algebra of smooth vector fields on the circle S^1 has a natural action as derivatives on the loop algebra, and the central extension of the algebra of smooth vector fields on the circle S^1 is in fact the above mentioned Virasoro algebra, which we will consider in the following section.

3.2 The Virasoro algebra.

The Virasoro algebra was probably first introduced by the physicist M.A. Virasoro in 1970 ([Vi]). It is a relatively nice infinite dimensional Lie algebra, which can be obtained as the central extension of the complexification of the smooth real vector fields on the unit circle S^1. Hence the Virasoro algebra can be viewed as the central extension of the complexification of the Lie algebra of the diffeomorphism group. However, the exponential mapping from the Lie algebra of real smooth vector fields on the unit circle to the diffeomorphism group is neither locally one-to-one nor onto (see [P-S, p. 28]).

It turns out that the orientation preserving diffeomorphism group on the unit circle $\mathit{Diff}^+(S^1)$ acts as a group of automorphisms on any loop group and that the orientation preserving subgroup $\mathit{Diff}^+(S^1)$ acts projectively on all the known representations of the loop groups (see [P-S, p. 5]). Finally we will mention that the Virasoro algebra plays an important role in theoretical physics, such as conformal field theory (see [B-P-Z 1]), string theory (see [C-T] and [L-T]) and statistical physics (see [B-P-Z 2]). We return to some of these questions in the end of this section. However, we will briefly explain how the representations of the Virasoro algebra (and $Vect(S^1)$) are relevant to physics. Yet, we emphasize that we are far from experts in dealing with these physical applications. Somehow physics is the science concerning with symmetries of nature. Since symmetries of a given physical system form a group, representation theory are of interest. In the study of crystals the relevant groups are the finite or discrete ones, for example certain translations with the distance given by the lattice length. In the study of atomic spectra it is the finite continuous groups, such as the group of rotations in three dimensions, $SO(3)$, which are used. Since space and time are (supposed) continuous, one expect continuous (Lie) groups to enter physics too. In quantum physics one usually expresses the symmetries naturally through a Lie group of Hermitian operators acting on an a priori Hilbert space, leaving the physical system (i.e. the Hamiltonian) invariant. Thereby one is led to consider the Lie algebra of generators of the Lie group. For example do the angular momenta generate $SO(3)$, hence they define an interesting Lie algebra. The simplest class of infinite dimensional Lie algebras, generalising the properties of finite dimensional Lie algebras naturally, are the affine Kac-Moody algebras and as mentioned in the former section, there is always associated with each affine Kac-Moody algebra a Virasoro algebra.

During the last twenty years an increasing number of physical models and theories have been presented using one or both of the above mentioned infinite dimensional algebras in an essential way. We will now give some explicit examples of physics where the Virasoro algebra enters. The first example appears in two-dimensional conformally invariant quantum physics, see for example [B-P-Z 1] or [L-T] where more details can be found. Introduce complex coordinates z and $\overline{z}$ in the two-dimensional Euklidian plane, $z = x + iy$ and $\overline{z} = x - iy$. The infinitesimal conformal transformations of the plane is given by the infinite number of generators $L_n = z^{n+1}\partial_z$ and $\overline{L}_n = \overline{z}^{n+1}\partial_{\overline{z}}$, where $\partial_z = \frac{d}{dz}$ and $\partial_{\overline{z}} = \frac{d}{d\overline{z}}$. In conformally invariant systems with periodic boundary conditions the generators L_n and $\overline{L}_n$ appears as coefficients of the Laurent expansion of the component of the energy-momentum tensor (see below). Symmetry and tracelessness of the energy-momentum tensor implies that it can be described by two independent components. In the $(z, \overline{z})$-coordinates we may choose $T = T'_{00} + T'_{01}$ and $\overline{T} = T'_{00} - T'_{01}$ as independent varibles, where the primes indicate that it is in the old coordinates. Then the conservation (or continuity) equation implies that $\partial_{\overline{z}}T = \partial_z\overline{T} = 0$, hence $T = T(z)$ and $\overline{T} = \overline{T}(\overline{z})$ are both holomorphic in z and $\overline{z}$, respectively. The energy-momentum tensor is then described in terms of the holomorphic components

$$T = T(z) \qquad \text{and} \qquad \overline{T} = \overline{T}(\overline{z})$$

which we may Laurent expand as

$$T(z) = \sum_{n\in\mathbf{Z}} z^{-n-2}L_n \qquad \text{and} \qquad \overline{T}(\overline{z}) = \sum_{n\in\mathbf{Z}} \overline{z}^{-n-2}\overline{L}_n$$

where

$$L_n = \frac{1}{2\pi i}\oint_{z=0} T(z)z^{n+1}\,dz \qquad \text{and} \qquad \overline{L}_n = \frac{1}{2\pi i}\oint_{\overline{z}=0} \overline{T}(\overline{z})\overline{z}^{n+1}\,d\overline{z}$$

The selfadjointness of T and $\overline{T}$ implies that $L_n^* = L_{-n}$ and $\overline{L_n^*} = \overline{L}_{-n}^*$ and it follows that each of the family of generators fulfils the commutation relations of the Virasoro algebra. In quantum field theory one usually describe the conformally invariant systems in terms of the correlation functions of the primary fields ϕ (i.e. fields fulfilling the particular transformation law, $\phi(z) \to f(\phi(z))^h\phi(f(z))$, under conformal transformations f in contrast to secondary fields). These fields satisfy $[L_n, \phi] = z^{n+1}\partial_z\phi + h(n+1)z^n\phi$ and the similar with $\overline{z}, \overline{L}_n$ and $\overline{h}$ instead of z, L_n and h, respectively, where h and $\overline{h}$ is the (energy) eigenvalue of L_0 and $\overline{L}_0$, respectively, corresponding to the eigenvector $\phi(0)\Omega$ (here Ω denotes the vacuum vector). Thus it can be shown that the two-point function, describing the field-correlation between point z_1 and z_2, becomes

$$\langle \Omega, \phi(z_1)\phi(z_2)\Omega \rangle = r^{-2(h+\overline{h})} e^{-2i\theta(h-\overline{h})} \langle \Omega, \phi(0)\phi(0)\Omega \rangle$$

where $z_2 - z_1 = re^{i\theta}$. The first factor $r^{-2(h+\overline{h})}$ comes from demanding rescaling (dilation) invariance and the second factor from demanding rotation invariance. The $h + \overline{h}$ is called the scaling dimension and is clearly related to the energy and $h - \overline{h}$ is refferred to as the conformal spin of the field ϕ. Hereby one reads the power law of the spatial separation, i.e. that the correlation function decay with separation as the separation to the power $-2(h + \overline{h})$. The next example is a continuation of the first. In the case of statistical physics, one often uses lattice models as follows. Consider a two-dimensional lattice with a spin variable at each vertex site, for example the Ising model (which is used to decribe such physical phenomena as feromagnetism of materials as ion). In a manageable approximation one supposes only interaction between nearest neighbours. At a certain critical point, for example a temperature point, the physical system can make a second order phase transition (i.e. the entropy is continuous as a function of temperature but its derivative is discontinuous), whereby it changes its physical state. It turns out that the lattice spacing becomes irrelevant, giving a scale invariant theory. In the conformally invariant case the theory, or model, is controlled by the Virasoro algebra, whose representation theory determines the critical exponents of the transition (see [B-P-Z 1]). As mentioned above the critical exponents specifies certain power law behaviours in spatial seperations, which is measurable in the laboratory. The final example concerns the way the Virasoro algebra enters the theory of strings through the diffeomorphism group (following [Se,G., p.336]). As a crude oversimplification one can say that one wants to describe unparametrized strings but finds it more convenient to describe parametrized strings. The group of diffeomorphisms act on the Hilbert space of states of parametrized strings by changing the parametrization. Following G. Segal [Se,G., p.336] a string is a one dimensional object, a mathematical curve on a manifold, i.e. it is the image of a smooth map ($x : [0, 1] \to \mathbf{R}^3$). When it moves it sweeps out a "wold-surface"

($x : [0,1] \times \mathbf{R} \to \mathbf{R}^4$) in Minkowski space-time. The points of which the string is made up are regarded as indistinguishable, so the parametrization has no physical significance (see also [Mi, chapter 9]).

The connection of the Virasoro algebra with the spin representation and the metaplectic representation will be shown for some particular cases in chapter 4 of applications.

In this section we will introduce and discuss the Virasoro algebra, with its applications in mind. We will in addition construct some representations of it, in the anti-symmetric Fock Hilbert space. This section is based on [G-W 2], [K-R], [P-S], [Jø] and [Mi], in that our preliminary discussion of the Virasoro algebra mostly follows that in [K-R, chapter 1]. [P-S, section 4.2] and [Jø, section 8.5] have been used as well. The subsequent discussion follows [Mi, section 7.4]. We will frequently refer to [G-W 2] and others.

Let $Vect(S^1)$ be the set of all smooth real vector fields on the circle S^1, i.e. $X \in Vect(S^1)$ has, in local coordinates, the form $X(\theta) = f(\theta)\frac{d}{d\theta}$ where f is a smooth real valued function on S^1 with period 2π. $Vect(S^1)$ is organized as a vector space by pointwise addition and natural multiplication by scalars (from $\mathbf{R}$). A direct calculation shows that the Lie bracket, also called the commutator, given by

$$[f(\theta)\frac{d}{d\theta}, h(\theta)\frac{d}{d\theta}] = (f(\theta)h'(\theta) - f'(\theta)h(\theta))\frac{d}{d\theta}$$

where $f'(\theta) = \frac{d}{d\theta}f(\theta)$ and $h'(\theta) = \frac{d}{d\theta}h(\theta)$, turns $Vect(S^1)$ into a Lie algebra (it is only the Jacobi identity which is not completely trivial).

A smooth function f on S^1 is always square integrable and a basis can be taken as $\{1, \cos n\theta, \sin n\theta\}_{n\in\mathbf{N}}$. So a basis over $\mathbf{R}$ for $Vect(S^1)$ is given by the vector fields $\frac{d}{d\theta}, \cos(n\theta)\frac{d}{d\theta}$ and $\sin(n\theta)\frac{d}{d\theta}$, $n \in \mathbf{N}$. Instead of considering the Lie algebra of smooth real vector fields on the circle S^1, we could consider the Lie algebra of smooth complex vector fields on the circle S^1, with basis given by the vector fields $e^{in\theta}\frac{d}{d\theta}$, $n \in \mathbf{Z}$.

This complex Lie algebra agrees with the complexification of $Vect(S^1)$, the complex linear span of (basis) vectors from $Vect(S^1)$. We denote this complex Lie algebra by $\mathfrak{d}$. It follows that we may consider $d_n = i \cdot e^{in\theta}\frac{d}{d\theta} = e^{in\theta}d_0$ as a basis for $\mathfrak{d}$. These basis elements satisfy the following commutation relations.

$$[d_m, d_n] = (m-n)\cdot d_{m+n}, \quad m, n \in \mathbf{Z}$$

The Lie algebra $Vect(S^1)$ can be considered as the Lie algebra of the group G of orientation preserving (real) diffeomorphisms of S^1, i.e.

$$G = Diff^+(S^1) = \left\{\gamma : \gamma(e^{i\theta}) = e^{i\phi(\theta)}, for some \phi \in \mathcal{M}\right\}$$

where $\phi \in \mathcal{M}$ means that ϕ is a smooth real 2π-periodic function with $\phi'(\theta) > 0$.

The product of two elements $\xi, \eta \in G$ is given by the composition of mappings $(\xi, \eta)(z) = \xi(\eta(z))$ for each $z = e^{i\theta} \in S^1$. We can define a representation π of G on the vector space of smooth complex valued functions f on S^1 by

$$\pi(\gamma)f(z) = f(\gamma^{-1}(z))$$

It is clearly a representation, since

$$\begin{aligned}
\pi(\gamma_1 \circ \gamma_2)f(z) &= f\left(\gamma_2^{-1}\left(\gamma_1^{-1}(z)\right)\right) \\
&= \left(f \circ \gamma_2^{-1}\right)\left(\gamma_1^{-1}(z)\right) \\
&= \left(\pi(\gamma_1)\left(f \circ \gamma_2^{-1}\right)\right)(z) \\
&= \pi(\gamma_1)\left(\pi(\gamma_2)f\right)(z) \\
&= \left(\pi(\gamma_1)\pi(\gamma_2)\right)f(z)
\end{aligned}$$

notice that $\pi(\gamma)$ is invertible for every $\gamma \in G$, since $\gamma^{-1} \in G$ and $\pi(\gamma)^{-1} = \pi(\gamma^{-1})$. We may write the d_n's in the z-koordinate, $z = e^{i\theta}$, as $d_n = -z^{n+1}\frac{d}{dz}$, $n \in \mathbf{Z}$. The Fourier expansion then gives

$$\epsilon(z) = \gamma(z) - z = \sum_{n\in\mathbf{Z}} \epsilon_n z^{n+1}$$

where ϵ_n denotes the $(n+1)$'th Fourier coefficient of $\epsilon(z)$. Normally one take γ close to the identity, whence

$$\gamma^{-1}(z) \simeq z - \sum_{n\in\mathbf{Z}} \epsilon_n z^{n+1}$$

where $\simeq$ means equal up to first order in the ϵ_n's. Then

$$\begin{aligned}\pi(\gamma)f(z) &\simeq f\left(z - \sum_{n\in\mathbf{Z}} \epsilon_n z^{n+1}\right) \\ &\simeq f(z) - \sum_{n\in\mathbf{Z}} \epsilon_n z^{n+1} \frac{d}{dz} f(z) \\ &= \left(1 + \sum_{n\in\mathbf{Z}} \epsilon_n d_n\right) f(z)\end{aligned}$$

This shows that the d_n's form a basis of the complexification of the Lie algebra of the group G.

In the following we will consider $Vect(S^1)$ as a subalgebra over $\mathbf{R}$ consisting of real elements of the complex Lie algebra $\mathfrak{d}$. The real elements of $\mathfrak{d}$ are those which are skew-invariant under the complex conjugation, which maps d_n into d_{-n}. That is, defining an anti-linear anti-involution α on $\mathfrak{d}$ by

$$\alpha(d_n) = d_{-n}$$

and

$$\alpha\left(\sum \lambda_n d_n\right) = \sum \overline{\lambda} \alpha(d_n)$$

where $\lambda_n \in \mathbf{C}$, then

$$\begin{aligned}\alpha([d_m, d_n]) &= (m-n)d_{-m-n} \\ &= -[d_{-m}, d_{-n}] \\ &= [\alpha(d_n), \alpha(d_m)]\end{aligned}$$

for all $x, y \in \mathfrak{d}$. Hence $Vect(S^1) \subset \mathfrak{d}$ consists of elements invariant under the action of $-\alpha$.

Let V be an arbitrary vector space with an Hermitian form $\langle\cdot,\cdot\rangle$, which in our case almost always will be an inner product. Consider a unitary representation of $G = Diff^+(S^1)$ on V. If we suppress the representation symbol and identify the elements in G with the corresponding operator on V, that is, we organize V as a G-module, then

$$\langle\gamma(u), \gamma(v)\rangle = \langle u, v\rangle$$

for any $u, v \in V$ and $\gamma \in G$. On the Lie algebra level, this means that

$$\langle x(u), v\rangle = -\langle u, x(v)\rangle$$

for $x \in Vect(S^1)$. Since $x = -\alpha(x)$ for any $x \in Vect(S^1)$, by construction of α, we see that

$$\langle x(u), v\rangle = \langle u, \alpha(x)(v)\rangle$$

for any $x \in Vect(S^1)$. In general we say that a Hermitian form $\langle\cdot,\cdot\rangle$ is contravariant if $\langle x(u), v\rangle = \langle u, \alpha(x)(v)\rangle$ for all x in a Lie algebra $\mathfrak{g}$ and all u and v in a representation space of $\mathfrak{g}$, where α is an anti-linear anti-involution on $\mathfrak{g}$.

In the case where the form is non-degenerate, this means that $x^* = \alpha(x)$ for all $x \in \mathfrak{g}$, where x^* denote the Hermitian adjoint with respect to the non-degenerated Hermitian form. We call the representation unitary if the contravariant Hermitian form, in addition, is positive definite.

In [K-R, p. 7] concrete irreducible representations of G on the density spaces are constructed.

It appears that some of these density spaces carries non-degenerated Hermitian contravariant forms and that all these representations are unitary. We will not make this construction here (a detailed treatment of this is given in [K-R, p. 7]), but rather consider the question of central extension of $\mathfrak{d}$, giving the Virasoro algebra.

The Lie algebra $\mathfrak{d}$ admits a non-trivial two-cocycle ω, namely

$$\omega(d_m, d_n) = \delta_{m+n} \cdot \frac{m(m^2-1)}{12}$$

where ω has been normalized to vanish on the $\mathfrak{sl}_2$-subalgebra spanned by $\{d_{-1}, d_0, d_1\}$. (The factor $\frac{1}{12}$ is a convention, see below).

By a direct calculation, the two-cocycle property follows:

$$\omega(d_k, [d_m, d_n]) + \omega(d_m, [d_n, d_k]) + \omega(d_n, [d_k, d_m]) = 0$$

In general for $X = f(e^{i\theta})\frac{d}{d\theta}$ and $Y = h(e^{i\theta})\frac{d}{d\theta}$ it reads

$$\omega(X, Y) = \frac{1}{24\pi i}\int_0^{2\pi} \left(f''(e^{i\theta}) + f(e^{i\theta})\right) h'(e^{i\theta}) d\theta$$

as a direct calculation shows.

with $\omega_m = \omega_{m,-m}$ and $\omega_0 = 0$. The anti-symmetry of the commutator gives that $\omega_{-m} = -\omega_m$. If we use the Jacobi identity for d_k, d_m and d_n with $k+m+n=0$, we get

$$\begin{aligned} 0 &= [d_k,[d_m,d_n]] + [d_m,[d_n,d_k]] + [d_n,[d_k,d_m]] \\ &= ((m-n)\omega_{m+n} - (2n+m)\omega_m + (2m+n)\omega_n)\cdot c \end{aligned}$$

for all $m,n \in \mathbf{Z}$. Putting $n=1$, this gives

$$(m-1)\omega_{m+1} = (m+2)\omega_m - (2m+1)\omega_1$$

for every $m \in \mathbf{Z}$. This is a linear recursion relation, and since ω_1 and ω_2 determinate all ω_m, for $m \geq 3$, and then all ω_m for $m \in \mathbf{Z}$, due to the fact that $\omega_0 = 0$ and $\omega_{-m} = -\omega_m$, the solution space is at most 2-dimensional.

Observing that $\omega_m = m$ and $\omega_m = m^3$ both are solutions, it follows that the general solution is given by $\omega_m = \alpha\cdot m + \beta\cdot m^3$. If $\beta = 0$, we may define the transformation $d_0 \to b_0 = d_0 + \frac{1}{2}\alpha\cdot c$ and $d_n \to b_n = d_n$, for $n \neq 0$. Then are $[b_0,b_n] = -nb_n$ and $[b_m,b_n] = (m-n)b_{m+n}$, which is an algebra without central charge, i.e. $c=0$ and ω_m is a coboundary. Hence the algebra for $\beta = 0$ is equivalent to $\mathfrak{d}$.

For a non-trivial central extension $\beta \neq 0$, we notice that the transformation $d_0 \to b_0 = d_0 + \frac{1}{2}(\alpha+\beta)\cdot c$ and $d_n \to b_n = d_n$, for $n \neq 0$, transform the commutation relations into $[b_m,b_n] = (m-n)b_{m+n} + \tilde{\omega}(b_m,b_n)\cdot c$ with the two-cocycle $\tilde{\omega}(b_m,b_n) = \delta_{m+n}\cdot\beta\cdot m(m^2-1)$. This transformation corresponds to a translation of the spectrum of d_0 by $\frac{1}{2}(\alpha+\beta)\cdot c$ in any representation (so it is bounded from below if d_0 were). Physically it is nothing but a translation of the energy scale and have therefore no physically significance (see below).

The factor α can therefore be chosen arbitrarily. Conventionally one selects $\alpha = -\beta$ such that $\omega_m = \beta(m^3 - m)$. By rescaling c, we can choose a fixed value for β. By convention $\beta = \frac{1}{12}$ is the value of choice.

Hence we arise with the above nontrivial two-cocycle

$$\omega(d_m,d_n) = \delta_{m+n}\cdot\frac{m(m^2-1)}{12}$$

The extended Lie algebra with basis $\{c, d_n : n \in \mathbf{Z}\}$, fulfilling the commutation relations

$$[d_n, c] = 0$$

and

$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m+n} \cdot \frac{m(m^2-1)}{12} \cdot c$$

is the most general central extension of $\mathfrak{d}$, and it is called the *Virasoro algebra*, denoted *Vir* . In fact we have proved above that every non-trivial central extension of the Lie algebra $\mathfrak{d}$ by a one-dimensional center is isomorphic to the Virasoro algebra.

Returning to the Sugawara construction discussed in the end of section 3.1, it follows at this point, that the operators L_n, $n \in \mathbf{Z}$ defined there, give us a representation of the Virasoro algebra with central charge given by $\frac{c \cdot \dim \mathfrak{g}}{Q+c}$, where this c is the value wrt the considered representation of the central element giving the extended loop algebra $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbf{C} \cdot c$.

It is also interesting to notice, that there is a natural action of $\mathfrak{d}$, and then of *Vir* by letting the central charge act trivially, as derivations on the loop algebra $\tilde{\mathfrak{g}}$, which lifts to $\hat{\mathfrak{g}}'$ by acting trivially on the central element, given by

$$d_k x(n) = n \cdot x(n+k)$$

for any $x \in g$, $k, n \in \mathbf{Z}$

It is interesting to consider representations of the Virasoro algebra, or equivalently projective representations of $\mathfrak{d}$ with non-trivial cocycle. let ρ_c be a representation of *Vir* , on a representation space V, with $\rho_c(c) = c \cdot I$, where c on the left hand side denote the central charge and the c on the right hand side denote a complex number.

Then we may consider a projective representation ρ of $\mathfrak{d}$ with commutation relations

$$[\rho(X), \rho(Y)] = \rho([X,Y]) + \omega(X,Y) \cdot c \cdot I$$

for $X, Y \in \mathfrak{d}$, where the commutators involved act on the respective Lie algebras, so the new one appearing on the left hand side is the common commutator of operators on V. The two representations are related by

$$\rho_c(X + \lambda \cdot c) = \rho(X) + \lambda \cdot c \cdot I$$

and

$$[X,Y]_c = [X,Y] + \omega(X,Y) \cdot c$$

Usually, one call ρ a highest weight representation if there exists a complex number h such that the operator $D_0 = \rho(d_0)$ diagonalizes on V with eigenvalues of the form $(h-n)$, $n \in \mathbf{N}\setminus\{0\}$, each of the corresponding eigenspaces V_{h-n} is finite-dimensional and such that $\dim V_h = 1$. The pair $(h, c) \in \mathbf{C}^2$, or sometimes just $h \in \mathbf{C}$, is called the highest weight of the highest weight representation ρ and we may choose a non-zero vector $v \in V_h$ to be named a heighest weight vector.

It is well known that for each $c \in \mathbf{C}$ and $h \in \mathbf{C}$ there exists an irreducible highest weight representation $\rho = \rho_{h,c}$, on $V = V(h, c)$ which is unique up to equivalence (see [G-W 2, p. 303] or [K-R, p. 24]).

Let L_n, $n \in \mathbf{N}$, denote the basis generators of a unitary representation of the Virasoro algebra. In physics L_0 usually denote the energy operator and is therefore required to be bounded from below (it is selfadjoint due to the unitarity of L_0 and so its possible eigenvalues are real). If ψ is an eigenvector of L_0 with eigenvalue λ, i.e. $L_0\psi = \lambda\psi$, then is $L_0 L_n\psi = (\lambda - n)L_n\psi$, for $n \in \mathbf{N}$, by use of the commutation relations for the Virasoro algebra. Hence $L_n\psi$ is an eigenvector of L_0 with eigenvalue $\lambda - n$ or $L_n\psi = 0$. Since the spectrum of L_0 is bounded from below there exist a lowest eigenvalue $h \in \mathbf{R}$ and a corresponding eigenvector v. Then is $L_0 v = hv$ and $L_n v = 0$ for all $n \in \mathbf{N}$. Thus we arrive with highest weight representations. In physics one seek unitary representations, if possible, and one focus on the irreducible ones (since such representation can be decomposed into a direct sum of irreducible ones, of which there are fewer and they are easier to handel). This is roughly the reason for seeking irreducible unitary highest weight representations and in fact it is the highest weight demand which put restrictions on the possible value of the central charge and the lowest energy eigenvalue. Let L_n be such a unitary highest weight representation of the Virasoro algebra with highest weight vector v and corresponding eigenvalue h, such that $L_0 v = hv$ and $L_n v = 0$, for $n \in \mathbf{N}$. Then

$$\begin{aligned} \|L_{-n}v\|^2 &= \langle v, L_n L_{-n} v\rangle = \langle v, [L_n, L_{-n}]v\rangle \\ &= \left(2nh + \frac{1}{12}n(n^2-1)c\right)\|v\|^2 \end{aligned}$$

wherefrom it follows that also c has to be real and in fact it follows that

$$2nh + \frac{1}{12}n(n^2-1)c \geq 0$$

for all $n \in \mathbf{N}$. Especially for $n = 1$ we get $h \geq 0$ and for n "large" we get $c \geq 0$. This condition is necessarily for the representation to be a unitary highest weight representation. However, it is not sufficient, since not all values of (c,h) in $[0,\infty[\ \times\ [0,\infty[$ provide us with unitary highest weight representations.

In the case $(h,c) \in \mathbf{R}^2$, $V = V(h,c)$ admits a unique (up to scalar multiple) non-zero contravariant Hermitian form. (see [G-W 2, p. 303]). It also follows that if, in addition h is non-negative and $c \geq 1$ then the highest weight representation is unitary (see [K-R, p. 26 and 27]).

We will not dwell at this point, but refer to [K-R] and [G-W 2]. However we will now consider the case $0 < c < 1$ briefly, using the Sugawara construction, and in the next section we will focus on the case $c = \frac{1}{2}$, by other methods. We return to the former case in section 3.4 where we treat it in all details by use of a generalization of the Sugawara construction.

Consider any admissible representation of an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$ with basis $\{x_i\}$, $i = 1, \ldots, \dim \mathfrak{g}$ such that $\{x_i\}$, $i = 1, \ldots, \dim \mathfrak{h}$ form a basis for a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}$ is simple.

The Lie algebra $\mathfrak{g}$ could for example be the extended loop algebra studied in section 3.1 (where we denoted it $\hat{\mathfrak{g}}$). Let $2Q_{\mathfrak{h}}$ and $2Q_{\mathfrak{g}}$ denote the eigenvalue of the Casimir operator of $\mathfrak{h}$ respectively $\mathfrak{g}$ in the adjoint representation of $\mathfrak{h}$ respectively $\mathfrak{g}$. We then define the Virasoro operators by the Sugawara construction for $\mathfrak{h}$ respectively $\mathfrak{g}$, i.e.

$$L_n^{\mathfrak{h}} = \frac{1}{Q_{\mathfrak{h}} + c} \sum_{i=1}^{\dim \mathfrak{h}} \sum_{k \in \mathbf{Z}} :x_i(k)x_i(n-k):$$

and

$$L_n^{\mathfrak{g}} = \frac{1}{Q_{\mathfrak{g}} + c} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{k \in \mathbf{Z}} :x_i(k)x_i(n-k):$$

Remark that that the operator c in the considered representation act as multiplication by a scalar c different from $-Q_{\mathfrak{h}}$ and $-Q_{\mathfrak{g}}$ such that we may divide by $Q_{\mathfrak{h}} + c$ and $Q_{\mathfrak{g}} + c$, respectively. Put $L_n = L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}}$ resulting in the following theorem.

Theorem 26 *The L_n's satisfy the Virasoro commutation relations with central charge $c = c_{\mathfrak{g}} - c_{\mathfrak{h}}$.*

Proof. This theorem can be found in [K-R, theorem 10.2] and [Mi, theorem 7.4.9], we follow their proofs. From the end of section 3.1 we know that

$$[L_n^{\mathfrak{g}}, x_i(m)] = -m \cdot x_i(m+n)$$

for $i = 1, \ldots, \dim \mathfrak{g}$ and

$$[L_n^{\mathfrak{h}}, x_i(m)] = -m \cdot x_i(m+n)$$

for $i = 1, \ldots, \dim \mathfrak{h}$ for all $n, m \in \mathbf{Z}$.

By subtraction we get $[L_n, x_i(m)] = 0$, for $i = 1, \ldots, \dim \mathfrak{h}$ and all $n, m \in \mathbf{Z}$. Then it follows that $[L_n, L_m^{\mathfrak{h}}] = 0$, for all $n, m \in \mathbf{Z}$, since the $L_m^{\mathfrak{h}}$'s are constructed from the $x_i(m)$, with $i = 1, \ldots, \dim \mathfrak{h}$, and

$$[L_n, L_m^{\mathfrak{h}}] = \frac{Q+c}{2} \sum_{k \in \mathbf{Z}} \sum_{i=1}^{\dim \mathfrak{h}} [L_n, x_i(k) x_i(m-k)] = 0$$

due to the fact that

$$[L_n, x_i(k) x_i(m-k)] = x_i(k) [L_n, x_i(m-k)] + [L_n, x_i(k)] \, x_i(m-k)$$

Writing $L_n^{\mathfrak{g}} = L_n^{\mathfrak{h}} - L_n$ the commutator $[L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}]$ splits into two pieces, so

$$\begin{aligned} [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] &= [L_n^{\mathfrak{h}}, L_m^{\mathfrak{h}}] + [L_n, L_m] \\ &= (n-m) L_{n+m}^{\mathfrak{h}} + \delta_{n+m} \cdot \frac{n(n^2-1)}{12} \cdot c_{\mathfrak{h}} + [L_n, L_m] \end{aligned}$$

and since

$$[L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] = (n-m) L_{n+m}^{\mathfrak{g}} + \delta_{n+m} \cdot \frac{n(n^2-1)}{12} \cdot c_{\mathfrak{g}}$$

it follows that

$$[L_n, L_m] = (n-m) L_{n+m} + \delta_{n+m} \cdot \frac{n(n^2-1)}{12} \cdot (c_{\mathfrak{g}} - c_{\mathfrak{h}})$$

implying that the L_n's satisfy the Virasoro commutation relations, with central charge $c = c_{\mathfrak{g}} - c_{\mathfrak{h}}$ ∎

At this point we will mention that the above considerations could be done for a finite dimensional semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_q$, where for $i = 1, \ldots, q$ the $\mathfrak{g}_i$'s are simple algebras. Hence we can construct a Virasoro algebra associated with $\mathfrak{g}$ from a highest weight representation of $\hat{\mathfrak{g}}_1 \oplus \cdots \oplus \hat{\mathfrak{g}}_q$, simply by taking the sum of the Virasoro algebras.

Define $L_n^{\mathfrak{g}}$ as

$$L_n^{\mathfrak{g}} = L_n^{\mathfrak{g}_1} + \cdots + L_n^{\mathfrak{g}_q}$$

then it follows that the terms in $L_n^{\mathfrak{g}}$ commute with each other, since the different algebras commute. Therefore the $L_n^{\mathfrak{g}}$'s fulfil the commutation relations of the Virasoro algebra, with central charge simply the sum of central charges (which may all be different from each other).

As an example take $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{su}(2)$, the Lie algebra consisting of all 2×2 traceless anti-Hermitian matrices, corresponding to the group $Su(2)$, of unitary 2×2 matrices with determinant 1. The dimension of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is 3.

Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ consisting of the diagonal, i.e. $(x, y) \in \mathfrak{h}$ if and only if $x = y$. Let $\hat{\mathfrak{g}}_1$ and $\hat{\mathfrak{g}}_2$ be the respective central extensions with central charge c_1 respectively c_2. Then the central charge of the algebra spanned by the L_n's in the theorem above, is given by

$$c = \frac{3 \cdot c_1}{2 + c_1} + \frac{3 \cdot c_2}{2 + c_2} - \frac{3 \cdot (c_1 + c_2)}{2 + (c_1 + c_2)}$$

since the sum of the first two terms is the central charge of $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$ and the third term is the central charge of $\hat{\mathfrak{h}}$ (we have chosen the eigenvalues of the Casimir operators to be 2, by normalizations). If we had chosen c_1 and c_2 to be $c_1 = k \in \{0, 1, 2, \ldots\}$ and $c_2 = 1$ then

$$c = 1 - \frac{6}{(k+2)(k+3)}$$

where $k = \mathbf{N} \cup \{0\}$. The serie of representations giving this serie of central charges will be constructed explicitely later on by other means (see section 3.4).

From this example it appears that we have constructed a series of representations of the Virasoro algebra with central charge c_k, forming a discrete series, $0 \leq c_k < 1$. In fact it is shown in [K-R, p. 129-138] that, the irreducible representations of *Vir* in the highest weight representation $V(h, c)$

is unitary if and only if c is of the above form and h is of the form

$$h_k^{(r,s)} = \frac{((k+3)r-(k+2)s)^2-1}{4(k+2)(k+3)}$$

where $r, s \in \mathbf{N} \cup \{0\}$ such that $1 \leq s \leq r \leq k+1$ and $k = 0, 1, 2, \ldots$ or if $c \geq 1$ and $h \geq 0$ (this formula will also be discussed later on in section 3.4). It appears that this result is very important in two-dimensional statistical mechanical models (see [B-P-Z 1]). To be more specific, in [B-P-Z 1, section 2] (and [B-P-Z 2, p.766-767]) the authors shows that the corservation of the stress-energy tensor, of some conformally invariant two-dimensional statistical system with critical points (described by massless quantum field theories), give rise to Virasoro operators (these are the coefficients appearing in the Laurent expansions of the two independent components of the stress-energy tensor), which satisfy the commutation relations of the Virasoro algebra, see also the sub-introduction in this section. Moreover, they discuss the so caled minimal theories (in section 6 and on p.769-776, respectively) and that the central charge c have to be on the discrete form. The first few values of c_m has been confirmed by physisist in the sence that $c_1 = \frac{1}{2}$, $c_2 = \frac{7}{10}$, $c_3 = \frac{4}{5}$ and $c_4 = \frac{6}{7}$ corresponds to the Ising model, the Tricritical Ising model, the 3-state Potts model and the Tricritical 3-state Potts model, repectively, which all are well-known models in statistical physics.

As a another application of the above theorem we'll mention the socalled quantum equivalence theorem, which states that if $c = c_{\mathfrak{g}} - c_{\mathfrak{h}} = 0$ then $L_n^{\mathfrak{g}} = L_n^{\mathfrak{h}}$ for all $n \in \mathbf{Z}$ (for a proof sc [Mi, p. 185]). The quantum equivalence theorem has important applications in string theory (see [Mi]).

In this section we have introduced the Virasoro algebra, shown some natural connections with the loop algebras and discussed representations theory of the Virasoro algebra. Furthermore we have endeavoured to do this in a self-contained manner. Finaly we repeat the wery interresting fact, both from a mathematical point of view and a physical point of view, that all the irreducible highest weight representations of the Virasoro algebra are known ([Mi, p.174]) and that these are unitary if and only if $c \geq 1$ and $h \geq 0$ ([K-R, p.88-89]) or $c = c_k = 1 - \frac{6}{(k+2)(k+3)}$, and $h \in \{h_k^{(r,s)} : r, s \in \mathbf{N} \cup \{0\}, 1 \leq s \leq r \leq k+1\}$, for $k \in \mathbf{N} \cup \{0\}$ ([K-R, p.129-138]).

In the following section we will consider some special representations of the Virasoro algebra with central charge $c = \frac{1}{2}$. To our knowledge the following constructions had not been discussed in details before, however, it is proposed by Kac and Raina in [K-R]. In section 3.4 we construct a seie of representations with central charge c running through the above serie, all c in $[0,1[$.

3.3 Representation of *Vir* with $c = \frac{1}{2}$.

In this section we will construct some explicit representations of the Virasoro algebra *Vir* , with central charge $c = \frac{1}{2}$. As representation space we will use the anti-symmetric Fock Hilbert space $\mathcal{F}_\wedge(\mathcal{H})$, as discussed in chapter 1.

To our knowledge these constructions have not been considered in details earlier. However, it is suggested by Kac and Raina in [K-R, p.29]. As basis material for the idea of this section we mention [P-S, section 9.2], [K-R, chapter 2 and 3 together with corollary 12.1], [G-W 2, section 1] and [Jø, definition 8.5.1] beyond chapter 1 and sections 3.1 and 3.2 in this thesis.

Consider the socalled *fermionic oscillator algebra* generated by

$$\{x_n : n \in \mathbf{Z}_q\}$$

where $\mathbf{Z}_q$ denote the set $\mathbf{Z} + q$, i.e. $n \in \mathbf{Z}_q$ if and only if $n - q \in \mathbf{Z}$, fulfilling the anti-commutation relations

$$[x_m, x_n]_+ = x_m x_n + x_n x_m = \delta_{m+n}$$

for $m, n \in \mathbf{Z}_q$. We will only consider $q = 0$ and $q = \frac{1}{2}$, giving the two essential different cases. In the case $q = 0$ we talk about the *Ramond sector* and in the case $q = \frac{1}{2}$ we talk about the *Neveu-Schwarz sector*. The essential difference between the two cases is, that in the Ramond sector, $q = 0$, the index $m = 0 \in \mathbf{Z}_0$ is allowed (as is the case for arbitrary $q \in \mathbf{Z}$), but in the Neveu-Schwarz sector, $q = \frac{1}{2}$, the index $m = 0 \notin \mathbf{Z}_q$ is disallowed (as is the case for arbitrary $q \notin \mathbf{Z}$). This is essential, since a possible element x_0, which appear only in the Ramond sector, is a distinguished element because of the anti-commutator relations. In fact, $2x_0 = [x_0, x_0]_+ = 1$ implies that $x_0 = \frac{1}{2}$.

In the following we will construct a representation in the fermionic Fock Hilbert space $\mathcal{F}_\wedge(\mathcal{H})$, modelled over a given Hilbert space $\mathcal{H}$, and thereby construct a representation of the Virasoro algebra with central charge $c = \frac{1}{2}$.

Let $\{e_n\}_{n \in \mathbf{N}_q}$ denote two orthonormal bases in the Hilbert space $\mathcal{H}$, where $\mathbf{N}_q = \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ for $q = 0$ and $\mathbf{N}_q = \mathbf{N}_{\frac{1}{2}} = \mathbf{N}_0 + \frac{1}{2}$ for $q = \frac{1}{2}$, i.e. $\mathbf{N}_q = \mathbf{N}_0 + q$. We define the representations (linear homomorphisms of the algebra into an algebra of linear operators) π_q by $\pi_q(x_n) = a(e_n)$ for $n \in \mathbf{Z}_q$ positive and $\pi_q(x_n) = a^*(e_{-n})$ for $n \in \mathbf{Z}_q$ negative. Here $a(\cdot)$ and $a^*(\cdot)$

denote repectively the annihilation and the creation operators in the Fock representation of the CAR-algebra, discussed in section 1.2 of chapter 1.

In the Ramond sector $q = 0$, we moreover define

$$\pi_0(x_0) = \frac{1}{\sqrt{2}}(a(e_0) + a^*(e_0))$$

For notational reasons we introduce the abbreviation $a_n = a(e_n)$ and $a_n^* = a^*(e_n)$, where $n \in \mathbf{N}_q$. By the anti-commutation relations of the CAR-algebra, $[a_m, a_n^*]_+ = \langle e_m, e_n\rangle \cdot I = \delta_{m-n} \cdot I$ and $[a_m, a_n]_+ = 0 = [a_m^*, a_n^*]_+$, it follows that the mappings indeed define homomorphisms and therefore representations of the fermionic oscillator algebras, conserving the anti-commutator relations, as they should.

We may define an anti-linear anti-involution α on the fermionic oscillator algebras by $\alpha(x_n) = x_{-n}$, whereever $n \in \mathbf{Z}_q$. It is consistent with the involution on the CAR-algebra,

$$\pi_q(x_n)^* = a_n^* = \pi_q(x_{-n}) = \pi_q(\alpha(x_n))$$

for $n \in \mathbf{Z}_q$ positive,

$$\pi_q(x_n)^* = (a_{-n}^*)^* = a_{-n} = \pi_q(x_{-n}) = \pi_q(\alpha(x_n))$$

for $n \in \mathbf{Z}_q$ negative and in the case of the Ramond sector

$$\pi_0(x_0)^* = \frac{1}{\sqrt{2}}(a_0 + a_0^*) = \pi_0(x_0) = \pi_0(\alpha(x_0))$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathcal{F}_\wedge(\mathcal{H})$. Since the inner product is a positive definite contravariant Hermitian form with respect to the Hilbert space adjoint, as the anti-linear anti-involution, the representations become unitary.

In the following we discard the representation symbols π_q and simply write x_k for $\pi_q(x_k)$, hence we identify a_k with x_k, a_k^* with x_{-k} for $k \in \mathbf{Z}_q$ positive and in the case of the Ramond sector $\frac{1}{\sqrt{2}}(a_0 + a_0^*)$ with x_0.

From another point of view, we merely rename the operators, calling a_k for x_k, a_k^* for x_{-k}, for $k \in \mathbf{Z}_q$ positive and in the Ramond sector $\frac{1}{\sqrt{2}}(a_0+a_0^*)$ for x_0. Notice that we eventually have $x_k^* = x_{-k}$. So we consider (a modification of) the Fock representation of the CAR-algebra and could have started our construction here. However, the above discussion provides somehow a better understanding of the construction, in view of our treatment in section 3.2 and the analogous construction for the bosonic oscillator or Heisenberg algebra (see for example [K-R, p. 12]).

In agreement with the anti-symmetry in $\mathcal{F}_\wedge(\mathcal{H})$ we define normal ordering $:x_j x_k:$ of $x_j x_k$ by

$$:x_j x_k: = \begin{cases} x_j x_k & \text{for } j \leq k \\ -x_k x_j & \text{for } j > k \end{cases}$$

for $j, k \in \mathbf{Z}_q$. Notice that this defines bounded operators on $\mathcal{F}_\wedge(\mathcal{H})$, since both the annihilation and creation operators are bounded on $\mathcal{F}_\wedge(\mathcal{H})$. Now for any $k \in \mathbf{Z}$ we define the Virasoro operators on $\mathcal{F}_\wedge(\mathcal{H})$ by

$$L_k = \frac{1}{2} \sum_{j \in \mathbf{Z}_q} j :x_{-j} x_{j+k}: + \delta_k \cdot \frac{1 - 2 \cdot q}{16}$$

where the last term also may be written as $\delta_k \cdot \delta_q \cdot \frac{1}{16}$ and $q = 0, \frac{1}{2}$ as usual. Notice that indices on the L_k's are always integers independent of q. It turns out that these Virasoro operators are unbounded on $\mathcal{F}_\wedge(\mathcal{H})$, for which reason we have to specify their domains.

The product basis vectors $e_{i_1} \wedge \ldots \wedge e_{i_n}$, $i_1 < \cdots < i_n$, $n \in \mathbf{N}$ (see section 1.1 of chapter 1), form together with Ω a basis of $\mathcal{F}_\wedge(\mathcal{H})$. Collect the basis product vectors with the same index sum in $\mathbf{M}_q$, where $\mathbf{M}_q$ is $\mathbf{N}_0$ for $q = 0$ and $\mathbf{N}_{\frac{1}{2}} \cup \mathbf{N}_0 \setminus \{1\}$ for $q = \frac{1}{2}$, i.e. for a given $m \in \mathbf{M}_q \setminus \{0\}$ put

$$B_m = \operatorname{span}\left\{ e_{i_1} \wedge \cdots \wedge e_{i_n} : \sum_{k=1}^{n} i_k = m, \ n \in \mathbf{N} \right\}$$

and for $m = 0$ put $B_0 = \operatorname{span}\{\Omega\}$ for $q = \frac{1}{2}$, and $B_0 = \operatorname{span}\{\Omega, e_0\}$ for $q = 0$.

Observe that the $B_m, m \in \mathbf{M}_q$, are mutually orthogonal and spans all of $\mathcal{F}_\wedge(\mathcal{H})$, moreover each B_m is finite dimensional by construction. So we have a *grading* of $\mathcal{F}_\wedge(\mathcal{H})$, as the Hilbert space direct sum of all B_m, $m \in \mathbf{M}_q$, i.e.

$$\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{m \in \mathbf{M}_q} B_m$$

Moreover we define $\mathcal{D}_0$ as the algebraic direct sum of the B_m, $m \in \mathbf{M}_q$, i.e.

$$\mathcal{D}_0 = \oplus_{alg} B_m$$

Then $\mathcal{D}_0$ is evidently a dense subspace of $\mathcal{F}_\wedge(\mathcal{H})$, which we call *the finite energy subspace* of $\mathcal{F}_\wedge(\mathcal{H})$, since it turns out below that vectors in $\mathcal{D}_0$ represent particles in $\mathcal{F}_\wedge(\mathcal{H})$ with finite energy.

We choose $\mathcal{D}_0$ as the common domain for all the L_k, $k \in \mathbf{Z}$. Though it is obvious that $\mathcal{D}_0 = \mathcal{D}_0(q)$ depends on q in the same way that $B_m = B_m(q)$ does. The reason for this choice of domain is that all the infinite sums appearing in the definition of the L_k's reduce to finite sums on $\mathcal{D}_0$. Notice in this connection that the normal ordering in the definition of L_k does not in general change the product for $k \neq 0$. In fact $: x_{-j}x_{j+k} := x_{-j}x_{j+k}$, for $j \geq -\frac{k}{2}$, and $: x_{-j}x_{j+k} := -x_{j+k}x_{-j} = x_{-j}x_{j+k} - \delta_k$ for $j < -\frac{k}{2}$, where we have used the anti-commutator relations. So for $k \in \mathbf{Z} \setminus \{0\}$ is

$$L_k = \frac{1}{2} \sum_{j \in \mathbf{Z}_q} j \cdot x_{-j}x_{j+k}$$

However for $k = 0$, the normal ordering contributes formally with an infinite sum, but on $\mathcal{D}_0$ it reduces to a finite sum.

For any F in $\mathcal{D}_0$, there exists a $M \in \mathbf{M}_q$ such that $F_m = 0$, for $m > M$, where $F = \oplus_{m \in \mathbf{M}_q} F_m$, with each $F_m \in B_m$. Then for any $k \in \mathbf{Z}$ will

$$\begin{aligned} L_k F &= \frac{1}{2} \left(\sum_{j \geq -\frac{k}{2}} j \cdot x_{-j}x_{j+k} - \sum_{j < -\frac{k}{2}} j \cdot x_{j+k}x_{-j} \right) F \\ &= \frac{1}{2} \sum_{-\frac{k}{2} \leq j \leq M-k} j \cdot x_{-j}x_{j+k}F - \frac{1}{2} \sum_{-M \leq j < -\frac{k}{2}} j \cdot x_{j+k}x_{-j}F \end{aligned}$$

since each F_m in $F = \oplus_{m \in \mathbf{M}_q} F_m$ may be written as a limit of linear combination of the basis product vector $e_{i_1} \wedge \ldots \wedge e_{i_n}$, with $\sum_{l=1}^{n} i_l = m \in \mathbf{M}_q$ made up of (one-particle) basis vectors e_{i_l} with indices $i_l \in \mathbf{N}_q$ less that or equal to m.

In the previous paragraphs we have used that $x_{j+k} = a(e_{j+k})$ annihilate, for $j > M - k$, and gives zero when applied to F_m, and that $x_{-j} = a(e_{-j})$ annihilate, for $j < -M$, and also gives zero when applied to F_m. Observe that $L_k : B_m \to B_{m+k}$ and then leaves $\mathcal{D}_0$ invariant. So we have shown that the formally infinite sums in the definition of the L_k, $k \in \mathbf{Z}$, reduce to finite sums on the dense invariant domain $\mathcal{D}_0$. In the special case where $k = 0$ we have

$$
\begin{aligned}
L_0 &= \frac{1}{2}\sum_{j\in\mathbf{N}_q} j\cdot x_{-j}x_j - \frac{1}{2}\sum_{-j\in\mathbf{N}_q} j\cdot x_jx_{-j} + \frac{1-2q}{16} \\
&= \sum_{j\in\mathbf{N}_q} j\cdot x_{-j}x_j + \frac{1-2q}{16}
\end{aligned}
$$

We will call L_0 *the energy operator*, since the eigenvalues for L_0 is given by $h_q + m$, with the corresponding eigenspace B_m, where $h_q = \frac{1-2q}{16}$. Observe that L_0 is unbounded, but bounded from below by h_q, since its spectrum is bounded form below. Notice further that $\mathcal{D}_0$ consists of physical states with finite energy, hence the name.

Our goal is to construct representations of the Virasoro algebra. If the L_k's fulfil the commutation relations of the Virasoro algebra, then we have some *positive energy representations* ($h_q \geq 0$) which are also *finite energy representations* on $\mathcal{D}_0$ (for general definitions see [P-S, p.171]).

Below we show that the L_k, $k \in \mathbf{Z}$, indeed fulfil the commutation relations of the Virasoro algebra.

Lemma 27 *The operators L_k, $k \in \mathbf{Z}$, fulfil*

$$
[x_m, L_k] = \left(m + \frac{1}{2}k\right) x_{m+k}
$$

on $\mathcal{D}_0$, for all $m \in \mathbf{Z}_q$.

Proof. From above it follows that all formally infinite sums appearing below reduce to finite sums on $\mathcal{D}_0$, and that the normal ordering in the definitions of the L_k, $k \in \mathbf{Z}$ only contributes for $k = 0$. For $k \in \mathbf{Z}\setminus\{0\}$ we have

$$
\begin{aligned}
[x_m, L_k] &= \frac{1}{2}\sum_{j\in\mathbf{Z}_q} j\cdot [x_m, x_{-j}x_{j+k}] \\
&= \frac{1}{2}\sum_{j\in\mathbf{Z}_q} j\left([x_m, x_{-j}]_+ x_{j+k} - x_{-j}[x_m, x_{j+k}]_+\right) \\
&= \frac{1}{2}\sum_{j\in\mathbf{Z}_q} j\left(\delta_{m-j}\cdot x_{j+k} - x_{-j}\cdot \delta_{j+m+k}\right) \\
&= \frac{1}{2}\left(m\cdot x_{m+k} + (m+k)\cdot x_{m+k}\right) \\
&= \left(m + \frac{1}{2}k\right)\cdot x_{m+k}
\end{aligned}
$$

where we have used the anti-commutator relations and the formula

$$[x_m, x_n x_l] = [x_m, x_n]_+ x_l - x_n [x_m, x_l]_+$$

for all allowed indices m, n and l. For $k = 0$ we get

$$\begin{aligned}
[x_m, L_0] &= \sum_{j \in \mathbf{N}_q} j \left([x_m, x_{-j}]_+ x_j - x_{-j} [x_m, x_j]_+ \right) \\
&= \sum_{j \in \mathbf{N}_q} (j \cdot \delta_{m-j} \cdot x_j - j \cdot x_{-j} \cdot \delta_{j+m}) \\
&= \sum_{j \in \mathbf{Z}_q} j \cdot x_j \cdot \delta_{m-j} \\
&= m \cdot x_m
\end{aligned}$$

where we moreover have used that the scalar term (times the unit operator) appearing in L_0 doesn't influence the commutator. ∎

We shall use the above lemma to prove the proposition below.

Proposition 28 *The L_m, $m \in \mathbf{Z}$, fulfil the commutator relations of the Virasoro algebra with central charge $c = \frac{1}{2}$, i.e.*

$$[L_m, L_n] = (m - n) \cdot L_0 + \delta_{m+n} \cdot \frac{m^3 - m}{24}$$

for all $m, n \in \mathbf{Z}$.

Proof. As in the above lemma all sums appearing below reduce to finite sums on $\mathcal{D}_0$. In the case of $q = 0$, all the indices on the e_j's will belong to $\mathbf{N}_0$ and all other indices belong to $\mathbf{Z}$, i.e. the indices on the x_j's and on the L_k's. In the case of $q = \frac{1}{2}$ all the indices on the e_j's will belong to $\mathbf{N}_{\frac{1}{2}} = \mathbf{N} - \frac{1}{2}$ and the indices on the x_j's will belong to $\mathbf{Z}_{\frac{1}{2}} = \mathbf{Z} - \frac{1}{2}$, but the indices on the L_k's will belong to $\mathbf{Z}$.

Remember that the normal ordering in $L_m = \frac{1}{2} \sum_{j \in \mathbf{Z}} j :x_{-j} x_{j+m}:$ doesn't contribute unless $m = 0$, since $:x_j x_{j+m}: = x_{-j} x_{j+m} - \delta_m$.

Since it is trivial that $[L_0, L_0] = 0$, we can treat the case of $m = 0$, for $n \in \mathbf{Z} \setminus \{0\}$, by $[L_0, L_n] = -[L_n, L_0]$. This gives that the case for $m = 0$

follows directly from the general case, since our treatment will not concern the normal ordering appearing in L_n.

For arbitrary $m \in \mathbf{Z} \setminus \{0\}$ and $n \in \mathbf{Z}$ will

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2} \sum_{j \in \mathbf{Z}_q} j \cdot (x_{-j}[x_{j+m}, L_n] + [x_{-j}, L_n] x_{j+m}) \\
&= \frac{1}{2} \sum_{j \in \mathbf{Z}_q} j \cdot \left(j + m + \frac{1}{2} n\right) x_{-j} x_{j+m+n} \\
&\quad + \frac{1}{2} \sum_{j \in \mathbf{Z}_q} j \cdot \left(-j + \frac{1}{2} n\right) x_{-j+n} x_{j+n}
\end{aligned}$$

Put $-j+n = -i$, such that $j = i+n$ and $j+m = i+m+n$, and write j instead of i, in the second sum, i.e. we make the transformation $j \longrightarrow j+n$, then

$$\begin{aligned}
&[L_m, L_n] \\
&= \frac{1}{2} \sum_{j \in \mathbf{Z}_q} \left(j \cdot \left(j + m + \frac{1}{2} n\right) - (j+n)\left(j + \frac{1}{2} n\right)\right) x_{-j} x_{j+m+n} \\
&= \frac{1}{2}(m-n) \sum_{j \in \mathbf{Z}_q} j \cdot x_{-j} x_{j+m+n} - \frac{n^2}{4} \sum_{j \in \mathbf{Z}_q} x_{-j} x_{j+m+n} \\
&= (m-n) \cdot L_{m+n} - (m-n) \frac{1}{2} \sum_{j \in I_q} j \cdot \delta_{m+n} - \frac{n^2}{4} \sum_{j \in \mathbf{Z}_q} x_{-j} x_{j+m+n}
\end{aligned}$$

where we in the last equality have used that $: x_{-j} x_{j+m+n} : = x_{-j} x_{j+m+n}$ for $j \geq -\frac{m+n}{2}$ and that $: x_{-j} x_{j+m+n} : = -x_{j+m+n} x_{-j} = x_{-j} x_{j+m+n} + \delta_{m+n}$ for $j < -\frac{m+n}{2}$.

Notice that we are only summing over a finite set I_q in the second term, since only finitely many terms in $\sum_{j \in \mathbf{Z}_q} j \cdot x_{-j} x_{j+m+n}$ are non-vanishing on $\mathcal{D}_0$. We have that

$$-(m-n) \frac{1}{2} \sum_{j \in I_q} j \cdot \delta_{m+n} = -\delta_{m+n} \cdot m \cdot \sum_{j \in I_q} j$$

Moreover the last sum also reduces to a finite sum. In fact using that $x_{-j} x_{j+m+n} + x_{j+m+n} x_{-j} = \delta_{m+n}$ and $\sum_{j \in \mathbf{Z}_q} x_{-j} x_{j+m+n} = \sum_{j \in \mathbf{Z}_q} x_{j+m+n} x_{-j}$

due to the transformation $j \to -(j+m+n)$, we get

$$\begin{aligned} \sum_{j\in\mathbf{Z}_q} x_{-j}x_{j+m+n} &= \frac{1}{2}\sum_{j\in\mathbf{Z}_q}(x_{-j}x_{j+m+n}+x_{j+m+n}x_{-j}) \\ &= \frac{1}{2}\sum_{j\in J_q}\delta_{m+n} = \delta_{m+n}\cdot\frac{1}{2}\mathrm{card}(J_q) \end{aligned}$$

where J_q is a finite subset of $\mathbf{Z}_q$, since $\sum_{j\in\mathbf{Z}_q} x_{-j}x_{j+m+n}$ reduces to a finite sum on $\mathcal{D}_0$. So the cardinality $\mathrm{card}(J_q)$ of J_q is finite. Hence for $m,n\in\mathbf{Z}$ we have

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n}\cdot c_m(q)$$

where $c_m(q)\in\mathbf{C}$ is determined by

$$c_m(q) = \langle\Omega, ([L_m, L_{-m}] - 2mL_0)\,\Omega\rangle$$

It follows immediately that $c_0(q)=0$ and $c_{-m}(q) = -c_m(q)$. Therefore we only need to calculate $c_m(q)$ for $m\in\mathbf{N}$.

Observe that

$$L_0\cdot\Omega = \sum_{j\in\mathbf{N}_q} j\cdot x_{-j}x_j\Omega + \frac{1-2q}{16}\cdot\Omega = \frac{1-2q}{16}\cdot\Omega$$

since $j\cdot x_j\Omega = 0$ for $j\geq 0$. Moreover for $m\in\mathbf{N}$ we have that

$$L_m\Omega = \frac{1}{2}\left(\sum_{j\geq-\frac{m}{2}} j\cdot x_{-j}x_{j+m} - \sum_{j<-\frac{m}{2}} j\cdot x_{j+m}x_{-j}\right)\Omega = 0$$

since $x_{j+m}\Omega = 0$ for $j\geq-\frac{m}{2}$ and $x_{-j}\Omega=0$ for $j<-\frac{m}{2}$. It follows that

$$c_m(q) = \|L_{-m}\Omega\|^2 - (1-2q)\cdot\frac{m}{8}$$

for $m\in\mathbf{N}$, where we have used that $L_m^* = L_{-m}$, which follows from

$$\left\langle G, \sum_{j\geq-\frac{m}{2}} x_{-j}x_{j+m}F\right\rangle = \left\langle\sum_{j\geq-\frac{m}{2}} x_{-j-m}x_jG, F\right\rangle = \left\langle\sum_{j\geq\frac{m}{2}} x_{-j}x_{j-m}G, F\right\rangle$$

and similarly

$$\left\langle G, \sum_{j<-\frac{m}{2}} x_{j+m}x_{-j}F\right\rangle = \left\langle\sum_{j<\frac{m}{2}} x_{j+m}x_{-j}G, F\right\rangle$$

on all $G, F \in \mathcal{D}_0$.

We now calculate $L_{-m}\Omega$ for $m \in \mathbf{N}$. We have

$$L_{-m}\Omega = \frac{1}{2}\left(\sum_{j \geq \frac{m}{2}} j \cdot x_{-j}x_{j-m} - \sum_{j < \frac{m}{2}} j \cdot x_{j-m}x_{-j}\right)\Omega$$

For $j > m$ is $x_{j-m}\Omega = 0$ and for $j < 0$ is $x_{-j}\Omega = 0$. Hence the sums become finite as they should and

$$L_{-m}\Omega = \frac{1}{2}\left(\sum_{\frac{m}{2} \leq j \leq m} j \cdot x_{-j}x_{j-m} - \sum_{0 \leq j < \frac{m}{2}} j \cdot x_{j-m}x_{-j}\right)\Omega$$

Transform j into $m - j (\in \mathbf{Z}_q)$ in the first sum, such that $-j \rightarrow j - m$, $j - m \rightarrow -j$ and the summation over j such that $\frac{m}{2} \leq j \leq m$ becomes a summation over j where $0 \leq j \leq \frac{m}{2}$. Hence we get

$$L_{-m}\Omega = \frac{1}{2}\sum_{0 \leq j < \frac{m}{2}} (m - 2j)x_{j-m}x_{-j}\Omega \tag{3.1}$$

In the case of m even, we should add the term $\frac{1}{2}x_{-\frac{m}{2}}x_{-\frac{m}{2}}\Omega = \frac{1}{2}e_{\frac{m}{2}} \wedge e_{\frac{m}{2}} = 0$, so it vanishes and therefore is disposed of. For further treatment we have to divide equation (3.1) into the two separate cases of the Ramond sector ($q = 0$) and the Neveu-Schwarz sector ($q = \frac{1}{2}$) and moreover subdivide these cases into the separate subcases of m even and m odd.

Consider first the case of the Ramond sector, where $q = 0$. Here we get

$$L_{-m}\Omega = \frac{1}{2}\sum_{0 \leq j < \frac{m}{2}} (m - 2j)\tilde{e}_{m-j} \wedge \tilde{e}_j$$

where $\tilde{e}_0 = \frac{1}{\sqrt{2}}e_0$ and $\tilde{e}_j = e_j$ for $0 < j < \frac{m}{2}$. The largest value j can obtain in the sum is different due to the parity of m. For m even $\frac{m}{2} - 1$ is the largest value of j allowed in the sum and for m odd $\frac{m-1}{2}$ is the largest value of j allowed in the sum. First, we will consider the subcase where m is even. Observing that the product vectors appearing in the sum are forming an orthogonal system with $\|\tilde{e}_m \wedge \tilde{e}_0\|^2 = \frac{1}{2}$ and each other normalized. Then

$$
\begin{aligned}
\|L_{-m}\Omega\|^2 &= \frac{1}{4}\sum_{j=1}^{\frac{m}{2}-1}(m-2j)^2+\frac{1}{4}\cdot\frac{m^2}{2} \\
&= \frac{1}{4}\sum_{j=1}^{\frac{m}{2}-1}(2j)^2+\frac{m^2}{8} \\
&= \sum_{j=1}^{\frac{m}{2}-1}j^2+\frac{m^2}{8}
\end{aligned}
$$

where we have used the transformation $j \to -j + \frac{m}{2}$. Now for $m \in \mathbf{N}$ even is

$$
\sum_{j=1}^{\frac{m}{2}-1} j^2 = \frac{m^3-3m^2+2m}{24}
$$

by induction. So

$$
\|L_{-m}\Omega\|^2 = \frac{m^3-3m^2+2m}{24}+\frac{m^2}{8} = \frac{m^3+2m}{24}
$$

for $m \in \mathbf{N}$ even. Hence

$$
c_m(0) = \|L_{-m}\Omega\|^2 - \frac{m}{8} = \frac{m^3-m}{24}
$$

for $m \in \mathbf{N}$ even.

Consider now the subcase where m is odd (still in the case where $q=0$). As above the product vectors, in the sum of $L_{-m}\Omega$, form an orthogonal system, each normalized, except for $j=0$, where $\|\tilde{e}_m \wedge \tilde{e}_0\|^2 = \frac{1}{2}$. Therefore we get

$$
\begin{aligned}
\|L_{-m}\Omega\|^2 &= \frac{1}{4}\sum_{j=1}^{\frac{m-1}{2}}(m-2j)^2+\frac{1}{4}\cdot\frac{m^2}{2} \\
&= \frac{1}{4}\sum_{j=1}^{\frac{m-1}{2}}(2j-1)^2+\frac{m^2}{8}
\end{aligned}
$$

due to the transformation $j \to -j + \frac{m+1}{2}$. Now for $m \in \mathbf{N}$ odd is

$$
\sum_{j=1}^{\frac{m-1}{2}}(2j-1)^2 = \frac{m^3-3m^2+2m}{6}
$$

by induction. So

$$\|L_{-m}\Omega\|^2 = \frac{1}{4} \cdot \frac{m^3 - 3m^2 + 2m}{6} + \frac{m^2}{8} = \frac{m^3 + 2m}{24}$$

for $m \in \mathbf{N}$ odd. Hence

$$c_m(0) = \|L_{-m}\Omega\|^2 - \frac{m}{8} = \frac{m^3 - m}{24}$$

for $m \in \mathbf{N}$ odd. Notice that the final formula for $c_m(0)$ is independent of the parity of m. In the Ramond sector ($q = 0$), we have

$$c_m(0) = \frac{m^3 - m}{24}$$

for all $m \in \mathbf{N}$, hence for all $m \in \mathbf{Z}$, since $c_0(0) = 0$ and $c_{-m}(0) = -c_m(0)$.

We now consider the Neveu-Schwarz sector, where $q = \frac{1}{2}$, which we treat in a similar way. Here we get

$$L_{-m}\Omega = \frac{1}{2} \sum_{0<j<\frac{m}{2}} (m - 2j)e_{m-j} \wedge e_j$$

since $j \in \mathbf{Z}_{\frac{1}{2}}$. Observing that the product vectors appearing in the sum form a orthonormal set. Then

$$\|L_{-m}\Omega\|^2 = \frac{1}{4} \sum_{0<j<\frac{m}{2}} (m - 2j)^2$$

As in the former case where $q = 0$ we split the further treatment of this case $q = \frac{1}{2}$ into two separate subcases - one for m even and one for m odd, since the largest value of j allowed in the sum depends of the parity of $m \in \mathbf{N}$.

Consider first the subcase of $m \in \mathbf{N}$ even. We get

$$\|L_{-m}\Omega\|^2 = \frac{1}{4} \sum_{j=\frac{1}{2}}^{\frac{m-1}{2}} (m - 2j)^2 = \frac{1}{4} \sum_{k=1}^{\frac{m}{2}} (2k - 1)^2$$

where $j \in \mathbf{N}_{\frac{1}{2}}$, but $k = \frac{m}{2} - \left(j - \frac{1}{2}\right) \in \mathbf{N}$. It is well-known that

$$\sum_{k=1}^{\frac{m}{2}} (2k - 1)^2 = \frac{1}{6}(m^3 - m)$$

by induction. Hence

$$c_m\left(\frac{1}{2}\right) = \|L_{-m}\Omega\|^2 = \frac{1}{4}\cdot\frac{1}{6}(m^3-m) = \frac{m^3-m}{24}$$

for $m \in \mathbf{N}$ even.

Consider now the subcase where m is odd. We have

$$\|L_{-m}\Omega\|^2 = \frac{1}{4}\sum_{j=\frac{1}{2}}^{\frac{m}{2}-1}(m-2j)^2 = \frac{1}{4}\sum_{k=1}^{\frac{m-1}{2}}(2k)^2 = \sum_{k=1}^{\frac{m-1}{2}}k^2$$

where $j \in \mathbf{N}_{\frac{1}{2}}$, but $k = \frac{m-1}{2} - \left(j-\frac{1}{2}\right) \in \mathbf{N}$. It easily follows that

$$\sum_{k=1}^{\frac{m-1}{2}}k^2 = \frac{m^3-m}{24}$$

by induction. Hence

$$c_m(\frac{1}{2}) = \|L_{-m}\Omega\|^2 = \frac{m^3-m}{24}$$

for $m \in \mathbf{N}$ odd.

As in the case where $q=0$ it turns out that the final formula for $c_m(\frac{1}{2})$ is independent of the parity of m. In the Neveu-Schwarz sector we have

$$c_m\left(\frac{1}{2}\right) = \frac{m^3-m}{24}$$

for all $m \in \mathbf{N}$, hence for all $m \in \mathbf{Z}$, since $c_0(\frac{1}{2}) = 0$ and $c_{-m}(\frac{1}{2}) = -c_m(\frac{1}{2})$. Observe furthermore that $c_m(0) = c_m(\frac{1}{2}) = c_m$ is independent of the sector considered, $q = 0, \frac{1}{2}$. That is, in both sectors we end up with the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n}\cdot\frac{m^3-m}{24}$$

for all $m,n \in \mathbf{Z}$. ∎

Hence we have constructed some positive and finite energy representations ρ_q of the Virasoro algebra $Vir = \oplus_{n\in\mathbf{Z}}\mathbf{C}\cdot d_n \oplus \mathbf{C}\cdot c$ in the anti-symmetric Fock Hilbert space, by

$$\rho_q(d_n) = L_n$$

where $q = 0, \frac{1}{2}$. However, it turns out that these representations of Vir are reducible. In fact, the orthogonal subspaces $\mathcal{F}_\wedge^+(\mathcal{H}) = \oplus_{n=0}^\infty(\wedge^{2n}\mathcal{H})$ and $\mathcal{F}_\wedge^-(\mathcal{H}) = \oplus_{n=0}^\infty(\wedge^{2n+1}\mathcal{H})$, consisting of respectively an even and an odd number of particles, are both invariant due to the fact that each L_k, $k \in \mathbf{Z}$, either create or annihilate two particles or conserve the number of particles. Hence each L_k, $k \in \mathbf{Z}$, conserves the parity of the number of particles in a given physical state. It turns out that the subrepresentations $\rho_q^\pm$ of the Virasoro algebra in $\mathcal{F}_\wedge^\pm(\mathcal{H})$, respectively, are irreducible, we return to this question below. Naturally the subrepresentations are still positive and finite energy representations, with the common domain

$$\mathcal{D}_0 \cap \mathcal{F}_\wedge^\pm(\mathcal{H}) = \mathcal{D}_0^\pm$$

for all the unbounded Virasoro operators $L_k, k \in \mathbf{Z}$.

We now modify the previously given definition of highest weight representations of Vir (following [G-W 2, p. 303] or [Jø, p. 238]) to lowest weight representations, trying to include our interesting cases. We call a representation ρ of $Vir = \oplus_{n\in\mathbf{Z}}\mathbf{C}\cdot d_n \oplus \mathbf{C}\cdot c$, with a dense domain $\mathcal{D}_\rho = \cap_{n\in\mathbf{Z}}\mathcal{D}(\rho(d_n))$ in a Hilbert space $\mathcal{K}$, a *lowest weight representation* if;

1. the operator $\rho(d_0)$ diagonalizes on $\mathcal{D}_\rho$ with eigenvalues of the form $h+n$, $n \in \mathbf{N}_0$ (except for a few $n \in \mathbf{N}$), and each of the corresponding eigenspaces V_{h+n} is finite dimensional;

2. especially $\dim V_h = 1$;

3. $\rho(c) = c \cdot I$, where c denotes both the central element and a complex number.

We remark that the bracket in demand 1 is also a modification relative to [G-W 2] and [Jø]. This modification is necessary in our case (see below). Usually the pair (h, c), and sometimes just h, is called the lowest weight and a non-zero normalized vector F_0 in V_h is called a lowest weight vector. The lowest weight vector is not necessarily cyclic. Observe that if ρ is unitary, as in our cases, since

$$\langle F, L_m G\rangle = \langle L_{-m}F, G\rangle$$

for all $m \in \mathbf{Z}$ and all $F, G \in \mathcal{D}_0$, then the lowest weight h is real, $h = \langle F_0, L_0F_0\rangle = \langle L_0F_0, F_0\rangle = \overline{h}$.

Consider first ρ_q^+, where $\mathcal{K} = \mathcal{F}_\wedge^+(\mathcal{H})$ and $\mathcal{D}_\rho = \mathcal{D}_0^+$. We have $L_m = \rho_q^+(d_m)$, $m \in \mathbf{Z}$. We have shown that L_0 diagonalizes on $\mathcal{D}_0^+$, with eigenvalues of the form $h_q + m$.

If $q = 0$ is $h_0 = \frac{1}{16}$, $\Omega \in \mathcal{F}_\wedge^+(\mathcal{H})$ and $L_0\Omega = h_0\Omega$, and $m \in \mathbf{N}_0 \setminus \{1,2\}$, since $e_1 \notin \mathcal{F}_\wedge^+(\mathcal{H})$ and $e_1 \wedge e_1 = 0$.

If $q = \frac{1}{2}$ is $h_{\frac{1}{2}} = 0$, again $\Omega \in \mathcal{F}_\wedge^+(\mathcal{H})$ and $L_0\Omega = h_{\frac{1}{2}}\Omega$, and $m \in \mathbf{N}_0 \setminus \{1\}$, since we have an even number of particles or none and $e_{\frac{1}{2}} \wedge e_{\frac{1}{2}} = 0$. The corresponding eigenspaces are finite dimensional, $V_{h+m} = B_m$, $m \in \mathbf{N}$.

In both cases Ω is a lowest weight vector and $V_h = \mathrm{span}\{\Omega\} = B_0 \cap \mathcal{F}_\wedge^+(\mathcal{H})$ is one dimensional.

Consider next ρ_q^-, where $\mathcal{K} = \mathcal{F}_\wedge^-(\mathcal{H})$ and $\mathcal{D}_\rho = \mathcal{D}_0^-$. Again, $L_0 = \rho_q^-(d_0)$ diagonalizes on $\mathcal{D}_0^-$, with eigenvalues of the form $h_q + m$.

If $q = 0$, is $h_0 = \frac{1}{16}$, $e_0 \in \mathcal{F}_\wedge^-(\mathcal{H})$ and $L_0 e_0 = h_0 e_0$, and $m \in \mathbf{N}_0$. Here e_0 is the lowest weight vector, $V_h = B_0 \cap \mathcal{F}_\wedge^-(\mathcal{H}) = \mathrm{span}\{e_0\}$, and $V_{h+m} = B_m$ for $m \in \mathbf{N}$. If $q = \frac{1}{2}$, is $h_{\frac{1}{2}} = \frac{1}{2}$, since $e_{\frac{1}{2}} \in \mathcal{F}_\wedge^-(\mathcal{H})$ and $L_0 e_{\frac{1}{2}} = h_{\frac{1}{2}} e_{\frac{1}{2}}$, and $m \in \mathbf{N}_0$. Here $e_{\frac{1}{2}}$ is the lowest weight vector, $V_h = B_{\frac{1}{2}} = \mathrm{span}\{e_{\frac{1}{2}}\}$ and $V_{h+m} = B_{m+\frac{1}{2}}$.

Hence it turns out that the representations $\rho_q^\pm$, $q = 0, \frac{1}{2}$, are lowest weight representations of the Virasoro algebra, with central charge $c = \frac{1}{2}$. Notice that all energies, except for the lowest are greater than one.

As promised earlier we now return to the question of showing the irreducibility of $\rho_q^\pm$, $q = 0, \frac{1}{2}$. It is possible but tedious to show directly that $L_{-i_n} \cdots L_{-i_2} \cdot L_{-i_1}$, $0 < i \leq \cdots \leq i_n$, applied to the actual lowest weight vectors, span the corresponding representation spaces. We will only sketch the proof here, since we are giving an alternative but more elegant proof below.

From $[L_0, L_k] = -k \cdot L_k$, $k \neq 0$, it follows that L_k gives zero, when applied to the lowest weight vector, for $k \in \mathbf{N}$ (otherweise its energy eigenvalue would be lower than the lowest weight, which is a contradiction). It also follows that the energy of $L_{-i_n} \cdots L_{-i_2} \cdot L_{-i_1}$ applied to the lowest weight vector is $h_q + \sum_{j=1}^n i_j$, where h_q denote the energy eigenvalue of the lowest weight vector. Now, direct calculations shows that $L_{-i_n} \cdots L_{-i_2} \cdot L_{-i_1}$, $0 < i \leq \cdots \leq i_n$, applied to the lowest weight vector gives a linear combination of product basis vectors of the form $e_{k_1} \wedge \cdots \wedge e_{k_l}$, such that $\sum_{j=1}^l k_j = \sum_{j=1}^n i_j = m$. Moreover, the product basis vectors span a dense set in B_m. Hence $L_{-i_n} \cdots L_{-i_2} \cdot L_{-i_1}$ applied to the lowest weight vector span a dense set in $\mathcal{F}_\wedge(\mathcal{H})$.

We now turn to the more elegant proof. Suppose that one of the representations ρ is reducible, then the corresponding representations space would have a invariant subspace not containing the actual lowest weight vector. This subspace then contains a singular vector, i.e. a vector of lowest energy of the form $h + m_n > 1$ (or equivalently a vector G_0 such that $L_m G_0 = 0$ for all $m \in \mathbf{N}$). This singular vector generates a unitary representation of Vir „ with central charge $c = \frac{1}{2}$ and lowest weight $h + m_n > 1$, contradicting the fact that unitary representations of Vir with central charge $c = \frac{1}{2}$ have lowest weight either 0, $\frac{1}{2}$ or $\frac{1}{16}$ (see [K-R, p. 139]), which we will discuss in section 3.4.

We summarize the above discussion in the following theorem.

Theorem 29 *The representations $\rho_q^{\pm} : d_n \to L_n$ of the Virasoro algebra in $\mathcal{F}_{\wedge}^{\pm}(\mathcal{H})$ with common domain $\mathcal{D}_0^{\pm}$, as constructed above, are irreducible unitary lowest weight representations of positive and finite energy of the Virasoro algebra, with central charge $c = \frac{1}{2}$.*

In the Ramond sector, $q = 0$, with lowest weight $h_0^+ = h_0^- = \frac{1}{16}$ and corresponding lowest weight vectors Ω and e_0 respectively.

In the Neveu-Schwarz sector, $q = \frac{1}{2}$, with lowest weight $h_{\frac{1}{2}}^+ = 0$ and $h_{\frac{1}{2}}^- = \frac{1}{2}$ and corresponding lowest weight vectors Ω and $e_{\frac{1}{2}}$ respectively.

Proof. A direct consequence of the above discussion. ∎

So we have constructed some nice representations of the Virasoro algebra in the anti-symmetric Fock Hilbert space, using the fermionic oscillator algebra. Usually one constructs representations using the bosonic oscillator algebra, corresponding to the symmetric Fock Hilbert space. In the next chapter of applications, we give some examples of representations of the Virasoro algebra, in both the symmetric and the anti-symmetric Fock Hilbert space, when discussing the diffeomorphisms group. This is not surprising in view of the socalled boson-fermion correspondance, to be discussed in section 4.4, see also [K-R, p. 49-64] or [Mi, p. 193 - 202] for futher details. However, we first turn to the construction of a serie of representations of the Virasoro algebra in the anti-symmetric Fock Hilbert space, in the next section, by other means.

3.4 Representations of *Vir* with $c_m \in [0,1[$.

There are briefly speaking three essentially different "algebraic" methods to construct unitary representations of the Virasoro algebra: The Sugawara construction, described in section 3.1 (giving representations with central charge c larger than or equal to 1), the "oscillator constructions", described in section 3.3 in the case of the fermionic oscillator algebra (giving representations with central charge $c = \frac{1}{2}$), and finally the Goddard-Kent-Olive construction, which generalize the Sugawara construction and which we will deal with in this section (giving representations with central charge c belonging to a discrete serie in $[0,1[$). Moreover, there are some "analytical" methods, described in chapter 4, to construct unitary representations of the Virasoro algebra (with central charge $c = 1$).

For an excelent, but purely algebraic treatment of the Goddard-Kent-Olive construction we refer to [K-R, chapter 4 and 9-12]. As in the other cases we intend to approach the method in a self-content way, dealing with representations in the Fock Hilbert spaces. More precisely of how to construct unitary highest weight representations of the Virasoro algebra in the tensor products of subspaces of some anti-symmetric Fock Hilbert spaces. This construction will explicitely make use of unitary highest weight representations of the Kac-Moody algebra of $\mathfrak{sl}_2$, the complex Lie algebra of traceless 2×2 matrices, which itself uses some explicitely constructed unitary highest weight representations of some particular infinite matrix Lie algebras together with there loop and Kac-Moody algebras.

This section is based on sections 3.1 and 3.2 together with the above mentioned part of [K-R].

Let the algebraic direct sum $V = \oplus_{alg}\text{span}\{v_i\}$ denote an infinite dimensional complex vector space with basis $\{v_i : i \in \mathbf{Z}\}$. Any vector $u \in V$ may be written as $u = (u_i)_{i\in\mathbf{Z}}$ with respect to the above basis, where only a finite number of the u_i's are non-zero, whereby we may identify V with $\mathbf{C}^\infty$. Let $\mathfrak{a}'_\infty$ denote the (Lie) algebra consisting of elements $a = (a_{i,j})_{i,j\in\mathbf{Z}}$ such that $a_{i,j} = 0$, whenever $|i-j| > N$ for some $N \in \mathbf{N}$. The elements $a = (a_{i,j})_{i,j\in\mathbf{Z}}$ are called infinite matrices and they have a finite number of non-zero diagonals, $(a_{i,i+k})_{i\in\mathbf{Z}}, k \in \mathbf{Z}$. By the ordinary matrix product, $ab = (\sum_{j\in\mathbf{Z}} a_{i,j}b_{j,k})_{i,k\in\mathbf{Z}}$, it follows that $ab \in \mathfrak{a}'_\infty$, whenever $a, b \in \mathfrak{a}'_\infty$. Notice that only a finite number of terms in the sum contribute, hence the sum is finite. With the ordinary matrix commutator as Lie product $\mathfrak{a}'_\infty$ becomes a Lie algebra. Let $\mathfrak{gl}_\infty$ denote the subalgebra of $\mathfrak{a}'_\infty$, consisting of elements a such that only a finite number of entries are non-zero. It is a Lie algebra with respect to the ordinary matrix commutator. We notice that the unit matrices $e_{i,j}$, $i,j \in \mathbf{Z}$, with 1 entry at the (i,j)'th position and zero entry elsewhere, form a basis for $\mathfrak{gl}_\infty$. Moreover

$$e_{i,j}e_{m,n} = \delta_{j-m}e_{i,n}$$

and

$$[e_{i,j}, e_{m,n}] = \delta_{j-m}e_{i,n} - \delta_{i-n}e_{m,j}$$

There is a natural action (representation) of $\mathfrak{a}'_\infty$ and of $\mathfrak{gl}_\infty$ on V in terms of the unit matrices, given by

$$e_{i,j}v_k = \delta_{j-k}v_i$$

Then the shift operator s_k on V, $k \in \mathbf{Z}$, given by

$$s_k v_i = v_{i-k}$$

may be written as

$$s_k = \sum_{i \in \mathbf{Z}} e_{i,i+k}$$

Therefore, s_k acts as the matrix with 1 entry on the k'th diagonal and zero elsewhere, hence s_k corresponds to an element in $\mathfrak{a}'_\infty$ which we denote s_k too. The elements s_k, $k \in \mathbf{Z}$, form a commutative subalgebra of $\mathfrak{a}'_\infty$, since $[s_k, s_l] = 0$, for $k, l \in \mathbf{Z}$. We also notice that $\{s_k : k \in \mathbf{Z}\}v_i = \{v_j : j \in \mathbf{Z}\}$, for any $i \in \mathbf{Z}$.

Let $\mathfrak{gl}_2$ denote the Lie algebra of all 2×2 matrices with complex entries and $\mathfrak{sl}_2$ the Lie subalgebra of all traceless complex 2×2 matrices. At this point we notice that one could easily consider the corresponding $n \times n$ matrices instead, for any $n \in \mathbf{N}$, and that the following considerations also holds in general. However, we will limit ourself to the case when $n = 2$, since it is sufficient for our purpose. Of course both the Lie algebras above have natural actions on $\mathbf{C}^2$.

Let $\widetilde{\mathfrak{gl}}_2$ and $\widetilde{\mathfrak{sl}}_2$ be the loop algebras, as defined in section 3.1, corresponding to $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$, repectively. If the matrix units for $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$ are denoted by $e_{i,j}$, $i, j \in \mathbf{Z}$ (they are 2×2 matrices, but they may be identified canonical with $e_{i,j}$, $i, j \in \mathbf{Z}$ in $\mathfrak{gl}_\infty$, why we use the same symbol) then

$$e_{i,j}(k) = e_{i,j} \otimes t^k$$

for $i, j = 1, 2$ and $k \in \mathbf{Z}$, form a basis for $\widetilde{\mathfrak{gl}}_2$, as $e_{i,j}$, $i, j = 1, 2$, for $\mathfrak{gl}_2$. The multiplication in $\widetilde{\mathfrak{gl}}_2$ is given by

$$e_{i,j}(k)e_{m,n}(l) = e_{i,j}e_{m,n} \otimes t^{k+l} = \delta_{j-m}e_{i,n}(k+l)$$

on basis elements, whereby $\widetilde{\mathfrak{gl}}_2$ forms an associative algebra. The Lie bracket on $\widetilde{\mathfrak{gl}}_2$ becomes

$$[e_{i,j}(k), e_{m,n}(l)] = \delta_{j-m}e_{i,n}(k+l) - \delta_{i-n}e_{n,j}(k+l)$$

as it should, due to section 3.1.

Consider now the natural action of $\mathfrak{gl}_2$ on $\mathbf{C}^2$ with standard basis $\{u_1, u_2\}$. Then the loop algebra $\widetilde{\mathfrak{gl}}_2$ acts naturally in $\mathbf{C}[t, t^{-1}]^2 \cong \mathbf{C}^2 \otimes \mathbf{C}[t, t^{-1}]$, where $\mathbf{C}[t, t^{-1}]$ as earlier denotes the ring of Laurent polynomials, as follows. The vectors

$$v_{2k+j} = u_j \otimes t^{-k}$$

form a basis of $\mathbf{C}[t, t^{-1}]^2$ over $\mathbf{C}$, where the index $i = 2k + j \in \mathbf{Z}$, since $k \in \mathbf{Z}$ and $j = 1, 2$. Hereby we identify $\mathbf{C}[t, t^{-1}]^2$ with $\mathbf{C}^\infty$. Since $e_{i,j}(k) = e_{i,j} \otimes t^k$ and $v_{2n+j} = u_j \otimes t^{-n}$ it follows that

$$e_{i,j}(k)v_{2n+j} = (e_{i,j}u_j) \otimes t^{k-n} = u_i \otimes t^{k-n} = v_{2(k-n)+i}$$

This action allow us to determine the corresponding matrix $\tau(a(t))$ in $\mathfrak{a}'_\infty$ of any element $a(t) \in \widetilde{\mathfrak{gl}}_2$. For the basis elements it is simply defined as

$$\tau(e_{i,j}(k)) = \sum_{n \in \mathbf{Z}} e_{2(n-k)+i, 2n+j}$$

where $e_{2(n-k)+i,2n+j}$ denotes the matrix units in $\mathfrak{a}'_\infty$ and $e_{i,j}$ those in $\mathfrak{gl}_2$ ($n, k \in \mathbf{Z}$ and $i, j = 1, 2$). For a general $a(t) \in \widetilde{\mathfrak{gl}}_2$ it follows that the corresponding matrix in $\mathfrak{a}'_\infty$ takes the block form

$$\tau(a(t)) = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & a_{-1} & a_0 & a_1 & & \\ & & a_{-1} & a_0 & a_1 & \\ & & & a_{-1} & a_0 & a_1 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

where $a(t) = \sum_{k\in\mathbf{Z}} a_k \otimes t^k$ and each $a_k \in \mathfrak{gl}_2$. Thus τ is an injective homomorphism of $\widetilde{\mathfrak{gl}}_2$ as an associative algebra and then also as a Lie algebra. The image of $a(t) = \sum_{k\in\mathbf{Z}} a_k \otimes t^k$ under τ is evidently a strictly upper triangular matrix if and only if $a_k = 0$, for $k \in -\mathbf{N}$ and a_0 strictly upper triangular. Moreover, if a^* denotes the usual Hermitian adjoint of the 2×2 matrix a and we define an anti-linear anti-involution ω on $\widetilde{\mathfrak{gl}}_2$ by

$$\omega(a(k)) = a^* \otimes t^{-k} = a^*(-k)$$

then it follows directly that

$$\tau(\omega(a(k))) = \tau(a(k))^*$$

where $\tau(a(k))^*$ denote the usual Hermitian adjoint of $\tau(a(k))$ in $\mathfrak{a}'_\infty$. Furthermore

$$\begin{aligned}
s_k = \tau\left((e_{1,2} + e_{2,1} \otimes t)^k\right) &= (\tau(e_{1,2}) + \tau(e_{2,1} \otimes t))^k \\
&= \left(\sum_{n\in\mathbf{Z}} e_{2n+1,2n+2} + \sum_{n\in\mathbf{Z}} e_{2(n-1)+2,2n+1}\right)^k \\
&= \left(\sum_{n\in\mathbf{Z}} (e_{2n+1,2n+2} + e_{2n,2n+1})\right)^k \\
&= \left(\sum_{n\in\mathbf{Z}} e_{n,n+1}\right)^k = s_1^k = s_k
\end{aligned}$$

Suppose for a while that we have a projective representation of $\mathfrak{a}'_\infty$ with two-cocycle $\alpha(a, b), a, b \in \mathfrak{a}'_\infty$. This can be made into a linear representation of the central extension $\mathfrak{a}_\infty$ of $\mathfrak{a}'_\infty$, i.e.

$$\mathfrak{a}_\infty = \mathfrak{a}'_\infty \oplus \mathbf{C} \cdot c$$

with c in the center of $\mathfrak{a}_\infty$ and the new bracket given by

$$[a, b] = [a, b]_0 + \alpha(a, b) \cdot c$$

where $[a, b]_0 = ab - ba$ now denotes the usual commutator in $\mathfrak{a}'_\infty$. Since $\widetilde{\mathfrak{gl}}_2$ can be realized as a subalgebra of $\mathfrak{a}'_\infty$ there is also a projective representation of $\widetilde{\mathfrak{gl}}_2$, by the restriction (we still suppose that there is one of $\mathfrak{a}'_\infty$). Then there is also a linear representation of the central extension $\widehat{\mathfrak{gl}}'_2 = \widetilde{\mathfrak{gl}}_2 \oplus \mathbf{C} \cdot c$, as a subalgebra of $\mathfrak{a}_\infty$. Moreover, since $\mathfrak{sl}_2$ is a Lie subalgebra of $\mathfrak{gl}_2$ we may define $\tilde{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}'_2$ as subalgebras of $\widetilde{\mathfrak{gl}}_2$ and $\widehat{\mathfrak{gl}}'_2$, respectively, in an obvious manner. Then we may consider $\widehat{\mathfrak{sl}}'_2$ as a subalgebra of $\mathfrak{a}_\infty$ too. We emphasize that $s_k = \tau\left((e_{1,2}(0) + e_{2,1}(1))^k\right) \in \tau(\widetilde{\mathfrak{gl}}_2)$. But $s_{2k+1} \in \tau(\tilde{\mathfrak{sl}}_2)$ and $s_{2k} \notin \tau(\tilde{\mathfrak{sl}}_2)$, since

$$(e_{1,2}(0) + e_{2,1}(1))^{2k} \cong \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}^{2k} = t^k \cdot I \cong I \otimes t^k \notin \tilde{\mathfrak{sl}}_2$$

and

$$(e_{1,2}(0) + e_{2,1}(1))^{2k+1} \cong t^k \cdot \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \in \tilde{\mathfrak{sl}}_2$$

This will be important later on. However, we first have to discuss the representations of $\mathfrak{a}'_\infty$ supposed above.

Let $\mathcal{F}_\wedge(\mathcal{H})$ denote the anti-symmetric Fock Hilbert space, where $\{e_i : i \in \mathbf{Z}\}$ denote an orthonormal basis for the one particle Hilbert space $\mathcal{H}$ (e_i with $i \in -\mathbf{N}$ describe the fermionic anti-particles).

Define operators $E_{i,j}, i, j \in \mathbf{Z}$, on $\mathcal{F}_\wedge(\mathcal{H})$ by

$$E_{i,j} = \begin{cases} a^*(e_i)a(e_j) & \text{, for } i, j \geq 0 \\ a^*(e_i)a^*(e_j) & \text{, for } i \geq 0 > j \\ a(e_i)a(e_j) & \text{, for } j \geq 0 > i \\ a(e_i)a^*(e_j) - \delta_{i-j} = -a^*(e_j)a(e_i) & \text{, for } 0 > i, j \end{cases}$$

where $a^*(e_k)$ and $a(e_k)$ denote the creation and annihilation operators, respectively, in the Fock representation, described in section 1.2. For each $i, j \in \mathbf{Z}$ is $E_{i,j}$ obviously well-defined, in fact, it is a partial isometry on $\mathcal{F}_\wedge(\mathcal{H})$.

The commutator $[E_{i,j}, E_{m,n}]$ can be calculated directly using the CAR. For $i, j, m, n \geq 0$ we get

$$\begin{aligned} E_{i,j}E_{m,n} &= a^*(e_i)a(e_j)a^*(e_m)a(e_n) \\ &= a^*(e_i)\left(\delta_{j-m} - a^*(e_m)a(e_j)\right)a(e_n) \\ &= \delta_{j-m}a^*(e_i)a(e_n) - a^*(e_m)a^*(e_i)a(e_n)a(e_j) \\ &= E_{i,n}\delta_{j-m} - a^*(e_m)\left(\delta_{i-n} - a(e_n)a^*(e_i)\right)a(e_j) \\ &= E_{i,n}\delta_{j-m} - E_{m,j}\delta_{i-n} + E_{m,n}E_{i,j} \end{aligned}$$

so $[E_{i,j}, E_{m,n}] = E_{i,n}\delta_{j-m} - E_{m,j}\delta_{i-n}$ follows. In completely the same manner one obtains exactly the samme commutator relations for all the other cases except for $i \geq 0 > j$ and $n \geq 0 > m$ and for $j \geq 0 > i$ and $m \geq 0 > n$. For these two exceptional cases one arise with

$$[E_{i,j}, E_{m,n}] = E_{i,n}\delta_{j-m} - E_{m,j}\delta_{i-n} \pm \delta_{j-m}\delta_{i-n}$$

where the $+$ sign holds for j (and m) non-negative and the $-$ sign holds for j (and m) negative. The results are obtained by calculations similar to those above, for example

$$\begin{aligned} E_{i,j}E_{m,n} &= a^*(e_i)a^*(e_j)a(e_m)a(e_n) \\ &= a^*(e_i)\left(\delta_{j-m} - a(e_m)a^*(e_j)\right) a(e_n) \\ &= \delta_{j-m}a^*(e_i)a(e_n) - a(e_m)a^*(e_i)a(e_n)a^*(e_j) \\ &= E_{i,n}\delta_{j-m} - a(e_m)\left(\delta_{i-n} - a(e_n)a^*(e_i)\right) a^*(e_j) \\ &= E_{i,n}\delta_{j-m} - E_{m,j}\delta_{i-n} - \delta_{j-m}\delta_{i-n} + E_{m,n}E_{i,j} \end{aligned}$$

for $i \geq 0 > j$ and $n \geq 0 > m$, whereof the desired commutation relations follows. Define α on $\mathfrak{a}'_\infty$ by bilinearity and

$$\begin{aligned} \alpha(e_{i,j}, e_{m,n}) &= \begin{cases} 1 & \text{, for } j = m \geq 0 \text{ and } i = n < 0 \\ -1 & \text{, for } j = m < 0 \text{ and } i = n \geq 0 \\ 0 & \text{, otherwise} \end{cases} \\ &= \delta_{j-m}\delta_{i-n}\left(\chi(-i)\chi(j+1) - \chi(-j)\chi(i+1)\right) \end{aligned}$$

on the matrix units, where χ denotes the indicator function for $\mathbf{N}$. Thereby $\alpha(\cdot,\cdot)$ defines a two-cocycle sometimes called the Kac-Peterson two-cocycle ([Mi, p 179]). Then we get a projective representation π of $\mathfrak{a}'_\infty$ in $\mathcal{F}_\wedge(\mathcal{H})$ by putting $\pi(e_{i,j}) = E_{i,j}$ together with linearity. Thus

$$[\pi(e_{i,j})\pi(e_{m,n})] = \pi\left([e_{i,j}, e_{m,n}]\right) + \alpha(e_{i,j}, e_{m,n}) \cdot I$$

Of couse we have to check that the extension of π to all of $\mathfrak{a}'_\infty$ is well-defined by linearity. Notice first that any element in $\mathfrak{a}'_\infty$ may be written as a finite linear combination of elements of the form

$$a_k = \sum_{i \in \mathbf{Z}} \lambda_i e_{i,i+k}$$

due to the fact that any element of $\mathfrak{a}'_\infty$ has a finite number of non-vanishing diagonals. The above $\lambda_i \in \mathbf{C}$ are arbitrary. So the extension becomes well-defined on $\mathfrak{a}'_\infty$ if there is a well-defined extension to elements of the form a_k. Observe that $\sum_{i\in\mathbf{Z}} \lambda_i E_{i,i+k}$ can be split into four sum-terms corresponding to $i \in \{j \in \mathbf{N} \cup \{0\} : j \geq -k\}$, $i \in \{j \in \mathbf{N} \cup \{0\} : j < -k\}$ which contribute only for $k < 0$, $i \in \{j \in -\mathbf{N} : j < -k\}$ and $i \in \{j \in -\mathbf{N} : j \geq -k\}$ which contribute only for $k > 0$. Each of these sum-terms reduces to a finite sum on Ω and on basis product vectors (where we utilize the term $-\delta_{i-j}$, for $i = j < 0$). Especially for $i = i + k < 0$ (i.e. $k = 0$) we get

$$\sum_{i\in\mathbf{Z}} \lambda_i E_{i,i}(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_{i\in\{j_1,\ldots,j_n\}} (-\lambda_i)(e_{j_1} \wedge \cdots \wedge e_{j_n})$$

on an arbitrary basis product vector (in this section we use the convention that $j_1 > \cdots > j_n$ for product vectors). Hence we may extend the representation π to all of $\mathfrak{a}'_\infty$ in a well-defined manner by linearity, since the formally infinite sum reduce to a finite sum.

As mentioned earlier we may turn the projective representation constructed above into a linear Lie algebra representation. Define the central extension $\mathfrak{a}_\infty = \mathfrak{a}'_\infty \oplus \mathbf{C} \cdot c$ of $\mathfrak{a}'_\infty$, where c denotes a central element, with the new bracket

$$[a, b] = [a, b]_0 + \alpha(a, b) \cdot c$$

where $[a, b]_0 = ab - ba$ now denotes the old bracket on $\mathfrak{a}'_\infty$ given by the ordinary commutator. We extend π from $\mathfrak{a}'_\infty$ to $\mathfrak{a}_\infty$ by putting $\pi(c) = 1$. Thus we have a Lie algebra representation of $\mathfrak{a}_\infty$ in $\mathcal{F}_\wedge(\mathcal{H})$ given by π, constructed above. The Hilbert space inner product $\langle\cdot,\cdot\rangle$then defines a contravariant Hermitian form, since $\langle\pi(a)F, G\rangle = \langle F, \pi(a^*)G\rangle$, for all $F, G \in \mathcal{F}_\wedge(\mathcal{H})$, where a^* denotes the earlier defined anti-linear anti-involution on $\mathfrak{a}'_\infty$. Thus is π a unitary representation, since the inner product is positively definite, by definition.

The charge operator Q is defined by $Q\Omega = 0$ and

$$Q(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \left(\sum_{i=1}^{n} j_i\right)(e_{j_1} \wedge \cdots \wedge e_{j_n})$$

on basis vectors. It extends to all of $\mathcal{D}$ by linearity, where $\mathcal{D}$ denote $\oplus_{alg} \wedge^n \mathcal{H}$ (it is even well-defined on $\mathcal{D}(N)$, where N denotes the number operator).

Considerations as those in section 1.5 shows that Q has spectrum $\mathbf{Z}$ and that $\mathcal{F}_\wedge(\mathcal{H})$ may be decomposed as

$$\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{q\in\mathbf{Z}}\mathcal{H}_q$$

where $\mathcal{H}_q$ denotes the eigenspace of Q corresponding to eigenvalue $q \in \mathbf{Z}$. Obviously, each $E_{i,j}$ conserves the charge of any basis product vector (or yields zero), hence it leaves each charge sector $\mathcal{H}_q$ invariant. Thereby $\pi_q = \pi|_{\mathcal{H}_q}$ defines a unitary representation of $\mathfrak{a}'_\infty$ in $\mathcal{H}_q$, for each $q \in \mathbf{Z}$.

Define $\mathbf{B}_0 = \mathrm{span}\{\Omega, e_0\}$ and

$$\mathbf{B}_m = \mathrm{span}\{e_{j_1} \wedge \cdots \wedge e_{j_n} : \sum_{i=1}^{n} |j_i| = m, n \in \mathbf{N}\}$$

for any $m \in \mathbf{N}$. Then we may decompose $\mathcal{F}_\wedge(\mathcal{H})$ as a Hilbert space direct sum

$$\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{m\in\mathbf{N}\cup\{0\}}\mathbf{B}_m$$

Moreover, we define $\mathcal{D}_0$ as the algebraic direct sum

$$\mathcal{D}_0 = \oplus_{alg}\mathbf{B}_m$$

Thus $\mathcal{D}_0$ is a dense subspace of $\mathcal{F}_\wedge(\mathcal{H})$. We now define an energy operator H on $\mathcal{D}_0$ by

$$HF_m = mF_m$$

for any $F_m \in \mathbf{B}_m$ and linearity. Notice that the spectrum of H indeed is $\mathbf{N} \cup \{0\}$ and that $\mathcal{F}_\wedge(\mathcal{H})$ decomposes into a direct Hilbert space sum of eigenspaces of H corresponding to eigenvalues $m \in \mathbf{N} \cup \{0\}$. Observe that $H\Omega = 0$ and

$$H(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \left(\sum_{i=1}^{n} |j_i|\right)(e_{j_1} \wedge \cdots \wedge e_{j_n})$$

on basis vectors. We define the sector vacuum Ω_q in each sector $\mathcal{H}_q$ to be a distinguished vector, namely the unique vector with lowest energy $m = \frac{1}{2}q(q-1)$, i.e.

$$\Omega_q = \begin{cases} e_{q-1} \wedge \cdots \wedge e_1 \wedge e_0 & , \text{for } q > 0 \\ e_{-1} \wedge e_{-2} \wedge \cdots \wedge e_q & , \text{for } q < 0 \\ \Omega & , \text{for } q = 0 \end{cases}$$

A direct calculation shows that

$$H\Omega_q = \frac{1}{2}q(q-1)\cdot\Omega_q$$

(the calculations are similar to those we will present in details later on, in section 4.2 in a particular case). Moreover, it follows by straight forward calculations that

$$E_{i_1,j_1}\cdot\ldots\cdot E_{i_n,j_n}\Omega_q$$

span a dense set in $\mathcal{H}_q$.

Below we will need the following decomposition of each $\mathcal{H}_q$ into sector-energy subspaces

$$\mathcal{H}_q = \oplus_{m\in\mathbf{N}\cup\{0\}}\mathcal{H}_q^{(m)}$$

where $\mathcal{H}_q^{(m)}$ denotes the eigenspace of the sector Hamiltonian $H|_{\mathcal{H}_q} - \frac{1}{2}q(q-1)$ corresponding to the sector-energy eigenvalue m, which equals the eigenspace of the energy operator corresponding to energy eigenvalue $m + \frac{1}{2}q(q-1)$, i.e.

$$\mathcal{H}_q^{(m)} = \mathcal{H}_q \cap \mathbf{B}_{m+\frac{1}{2}q(q-1)}$$

for each $q \in \mathbf{Z}$.

We know that each $\mathcal{H}_q$ is invariant under $\pi(\mathfrak{a}'_\infty)$ and that $\pi(\mathfrak{a}'_\infty)\Omega_q$ form a dense set in $\mathcal{H}_q$. Then it follows that π is irreducible. Suppose that U is an invariant subspace of $\mathcal{H}_q$ then $\mathcal{H}_q = U \oplus U^\perp$, where the orthogonal complement $U^\perp$ of U is taken in $\mathcal{H}_q$. Then the decomposition $\mathcal{H}_q = U \oplus U^\perp$ also respects the sector-energy decomposition which can be proved quite similarly to the proof of theorem 22 part 3, i.e. we may write

$$U = \oplus_{m\in\mathbf{N}\cup\{0\}}U_m \qquad \text{and} \qquad U^\perp = \oplus_{m\in\mathbf{N}\cup\{0\}}U_m^\perp$$

where $U_m = U \cap \mathcal{H}_q^{(m)}$ and $U_m^\perp = U^\perp \cap \mathcal{H}_q^{(m)}$. Since $\mathcal{H}_q^{(0)} = \mathrm{span}\{\Omega_q\}$ is one-dimensional, Ω_q belongs to either U_0 or $U_0^\perp$, say $\Omega_q \in U_0$, but Ω_q generates a dense set in $\mathcal{H}_q$ under $\pi(\mathfrak{a}'_\infty)$. Due to the invariancy of $U = \oplus_{m\in\mathbf{N}\cup\{0\}}U_m$ it follows that $U = \mathcal{H}_q$ and $U^\perp = \{0\}$. Hence $\pi_q = \pi|_{\mathcal{H}_q}$ is irreducible.

Observe that

$$E_{i,j}\mathcal{H}_q^{(m)} \subset \mathcal{H}_q^{(m+i-j)}$$

which easily follows by checking the four possible cases of $E_{i,j}$ directly. This inspires us to define the degree of $e_{i,j} \in \mathfrak{gl}_\infty$ as

$$\deg(e_{i,j}) = i - j$$

Futhemore, we define the degree of an element $a \in \mathfrak{gl}_\infty$ to be $\deg(a) = k$ if a is of the form $a = \sum \lambda_i e_{i,i+k}$, where the sum is over a finite subset of $\mathbf{Z}$. We may decompose $\mathfrak{gl}_\infty$ as a direct sum of homogeneous components $\mathfrak{g}_k$, i.e. subspaces of elemenets in $\mathfrak{gl}_\infty$ with fixed degree $k \in \mathbf{Z}$, as $\mathfrak{gl}_\infty = \oplus_{k\in\mathbf{Z}}\mathfrak{g}_k$. It then follows that

$$\pi(\mathfrak{g}_k)\mathcal{H}_q^{(m)} \subset \mathcal{H}_q^{(m+k)}$$

and that

$$\pi(\mathfrak{g}_k)\Omega_q = 0$$

whenever $k < 0$, since $E_{i,j}\Omega_q = 0$, for $i < j$. Since $E_{i,j} = \pi(e_{i,j}) \in \pi(\mathfrak{g}_{j-i})$ it follows that

$$\sum_{\substack{k_1+\cdots+k_n=m \\ k_1,\ldots,k_n\in\mathbf{N}\cup\{0\}}} \pi_q(\mathfrak{g}_{k_1})\cdot\ldots\cdot\pi_q(\mathfrak{g}_{k_n})\Omega_q$$

form a dense set in $\mathcal{H}_q^{(m)}$.

Put

$$\mathfrak{n}_+ = \oplus_{k<0}\mathfrak{g}_k$$

such that any $a \in \mathfrak{n}_+$ is a strictly upper triangular matrix in $\mathfrak{gl}_\infty$. Then

$$\pi_q(\mathfrak{n}_+)\Omega_q = 0$$

and

$$\pi_q(e_{i,i})\Omega_q = E_{i,i}\Omega_q = \lambda_i\Omega_q$$

with

$$\lambda_i = \begin{cases} 1 & \text{, for } 0 \le i \le q-1 \quad (q > 0) \\ -1 & \text{, for } q \le i < 0 \quad (q < 0) \\ 0 & \text{, otherwise} \end{cases} \tag{3.2}$$

Summarizing the above we get

Theorem 30 *For each $q \in \mathbf{Z}$ do π_q defines an irreducible unitary highest weight representation of $\mathfrak{gl}_\infty$ in $\mathcal{H}_q$ with the sector vacuum vector Ω_q as the highest weight vector and $\lambda(q) = \{\lambda_i : i \in \mathbf{Z}\}$, where λ_i is as in formula* (3.2)*, as the highest weight.*

Proof. Notice that this definition of a highest weight representation follows that in [K-R, p 41]; a collection of numbers $\lambda = \{\lambda_i : i \in \mathbf{Z}\}$ is called a highest weight of an irreducible representation π_λ in a vector space $\mathcal{L}(\lambda)$, called a highest weight representation, if there exists a non-zero vector ν_λ in $\mathcal{L}(\lambda)$, called a highest weight vector, such that

$$\pi_\lambda(\mathfrak{n}_+)\nu_\lambda = 0$$

and

$$\pi_\lambda(e_{i,i})\nu_\lambda = \lambda_i\nu_\lambda$$

We note that $\mathcal{L}(\lambda)$ is determined by λ. Putting $\lambda = \lambda(q)$, $\mathcal{L}(\lambda) = \mathcal{H}_q$, $\nu_\lambda = \Omega_q$, which is allowed due to the unique correspondance between q and $\{\lambda_i : i \in \mathbf{Z}\}$, given in (3.2). Thus it follows directly from above that π_q defines an irreducible highest weight representation of $\mathfrak{gl}_\infty$ in $\mathcal{H}_q$ with highest weight vector Ω_q and highest weight $\lambda(q)$. The unitarity of π_q follows, since the representation π_q of $\mathfrak{a}'_\infty$ is unitary. ∎

The tensor product of two irreducible unitary highest weight representations with highest weight vectors ν_1 and ν_2, respectively, gives a irreducible unitary highest weight representation in the *highest component* of the tensor product of the representation spaces, i.e. in the vector space generated by the highest weight vector $\nu_1 \otimes \nu_2$. Therefore we may construct irreducible unitary highest weight representations π_λ of $\mathfrak{gl}_\infty$ in a vector space $\mathcal{L}(\lambda)$ with highest weight $\lambda = k_1 \cdot \lambda(q_1) + \cdots + k_n \cdot \lambda(q_n)$, and highest weight vector $\nu_\lambda = \Omega_{q_1}^{\otimes k_1} \otimes \cdots \otimes \Omega_{q_n}^{\otimes k_n}$, where $\Omega_{q_j}^{\otimes k_j} = \Omega_{q_j} \otimes \cdots \otimes \Omega_{q_j}$, k_j times, $j = 1, \ldots, n$, $k_1, \ldots, k_n \in \mathbf{N}$, $q_1, \ldots, q_n \in \mathbf{Z}$ and $n \in \mathbf{N}$.

Returning for a while to the commutative subalgebra spanned by the shift operators in $\mathfrak{a}'_\infty$ we have

$$[\pi_q(s_n), \pi_q(s_k)] = \alpha(s_n, s_k) \cdot I$$

where

$$\begin{aligned}
\alpha(s_n, s_k) &= \sum_{i\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} \alpha(e_{i,i+n}, e_{j-k,j}) \\
&= \sum_{i\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} \delta_{i-j}\delta_{i+n-(j-k)}\left(\chi(-i)\chi(i+n+1) - \chi(i+1)\chi(-i-n)\right) \\
&= \sum_{-n\leq i<0} \delta_{n+k} - \sum_{0\leq i<-n} \delta_{n+k} \\
&= n \cdot \delta_{n+k}
\end{aligned}$$

One recognises this as the commutation relations of the fermionic oscillator algebra, also known as the Heisenberg algebra (see for example [K-R, p 12]). Furthermore

$$\pi_q(s_0) = \sum_{i\in\mathbf{Z}} \pi_q(e_{i,i}) = \sum_{i\in\mathbf{Z}} E_{i,i} = q \cdot I$$

when applied to any vector in $\mathcal{H}_q$ and

$$\pi_q(s_k)\Omega_q = 0$$

for $k > 0$. Moreover, the vectors

$$\pi_q(s_{-k_1}) \cdot \ldots \cdot \pi_q(s_{-k_n})\Omega_q$$

where $k_1 \geq \cdots \geq k_n > 0$, are linearly independent and those with $\sum_{j=1}^n k_j = m$ belongs to $\mathcal{H}_q^{(m)}$, in fact they do form a basis for $\mathcal{H}_q^{(m)}$. This can be seen as follows. There are obviously $p(m)$ linearly independent vectors of the given form in $\mathcal{H}_q^{(m)}$, where $p(m)$ denote the number of partitions of $m \in \mathbf{N}$ into a sum of positive integers $\sum_{i=1}^n k_i = m$. But $p(m)$ is the dimension of $\mathcal{H}_q^{(m)}$ (this is shown in details in the proof of theorem 34 in section 4.4, in a similar case). So there are $\dim(\mathcal{H}_q^{(m)})$ numbers of linearly independent vectors and thus they form a basis of $\mathcal{H}_q^{(m)}$. Then π_q is an irreducible representation of the commutative subalgebra, since each $\mathcal{H}_q$ is indeed invariant under $\{\pi_q(s_k) : k \in \mathbf{N}\}$ and the linearly independent vectors $\pi_q(s_{-k_1}) \cdot \ldots \cdot \pi_q(s_{-k_n})\Omega_q$, with $k_1 \geq \cdots \geq k_n > 0$ span a dense set of $\mathcal{H}_q$, hence the irreducibility follows by a proof similar to that used in the case of $\mathfrak{a}'_\infty$ earlier.

We now return to the the loop algebras. Following the approach of section 3.1 we may, as mentioned earlier, extend the loop algebra $\mathfrak{gl}_2$ with a central element c giving

$$\widetilde{\mathfrak{gl}}'_2 = \widetilde{\mathfrak{gl}}_2 \oplus \mathbf{C} \cdot c$$

with respect to the Kac-Peterson two-cocycle, which becomes

$$\alpha\left(\tau(e_{i,j}(k)), \tau(e_{m,n}(l))\right) = \delta_{i-n}\delta_{j-m}\delta_{k+l} \cdot k$$

By linearity we get

$$\alpha\left(\tau(a(k)), \tau(b(l))\right) = \delta_{k+l} \cdot k \cdot \mathrm{Tr}(ab)$$

for $a, b \in \mathfrak{gl}_2$ arbitrarily, and for general $a(t), b(t) \in \mathfrak{gl}_2$ we get

$$\alpha\left(\tau(a(t), \tau(b(t))\right) = \mathrm{res}_0\left(\mathrm{Tr}(a'(t)b(t))\right)$$

where $a'(t)$ denote the derivative of $a(t)$ and res_0 the residue at $t = 0$.

The Lie bracket on $\widehat{\mathfrak{gl}}_2'$ is then given by

$$[a(k), c] = 0$$

and

$$[a(k), b(l)] = [a, b]_0(k + l) + k \cdot \delta_{k+l} \cdot \mathrm{Tr}(ab) \cdot c$$

for arbitrary $a(k), b(l) \in \widetilde{\mathfrak{gl}}_2$ and where $[a, b]_0 = ab - ba$, as earlier.

The Killing form $(a, b)_0 = \mathrm{Tr}(ab)$ on $\mathfrak{gl}_2$ gives a bilinear form on $\widetilde{\mathfrak{gl}}_2$ by

$$(a(k), b(l)) = \delta_{k+l}\mathrm{Tr}(ab)$$

which extend to all of $\widetilde{\mathfrak{gl}}_2$ by linearity as

$$(a(t), b(t)) = \mathrm{res}_0\left(t^{-1}\mathrm{Tr}(a(t)b(t))\right)$$

It is evidently a symmetric, invariant, non-degenerated bilinear form on $\widetilde{\mathfrak{gl}}_2$, which is verified by direct calculations. The form extends to $\widehat{\mathfrak{gl}}_2'$ by putting $(c, \widetilde{\mathfrak{gl}}_2) = 0$ and $(c, c) = 0$. On $\widehat{\mathfrak{gl}}_2'$ the form is of course degenerated, but it is still symmetric, invariant and bilinear. As in section 3.1 we therefore extend $\widehat{\mathfrak{gl}}_2'$ further, by a generator d, as

$$\widehat{\mathfrak{gl}}_2 = \widehat{\mathfrak{gl}}_2' \oplus \mathbf{C} \cdot d$$

with the old commutator on $\widehat{\mathfrak{gl}}_2'$ together with

$$[d, c] = 0$$

and

$$[d, a(k)] = k \cdot a(k)$$

giving an affine Kac-Moody algebra. It follows that $\widehat{\mathfrak{gl}}_2$ carries a non-degenerated, symmetric, invariant bilinear form $(\cdot, \cdot)$ given as in proposition 20 of section 3.1, with $s = 0$ (and $(a(t), b(t))_0 = \mathrm{Tr}(a(t)b(t))$).

Recall that $\widetilde{\mathfrak{gl}}_2$ is a subalgebra of $\mathfrak{a}'_\infty$ and that the anti-linear anti-involution ω on $\widetilde{\mathfrak{gl}}_2$ coincide with the one induced from $\mathfrak{a}'_\infty$. If we define $\omega(c) = c$, then the central extension $\widehat{\mathfrak{gl}}_2'$ of $\widetilde{\mathfrak{gl}}_2$ is a subalgebra of the central extension

$\mathfrak{a}_\infty$ of $\mathfrak{a}'_\infty$ too. We define $\tau(c) = c$. Since $s_k = \tau((e_{1,2}(0)+e_{2,1}(1))^k) \in \tau(\widehat{\mathfrak{gl}}_2')$ and π_q remains irreducible when restricted to the subalgebra spanned by the s_k, $k \in \mathbf{Z}$, it follows that $\pi_q(\tau(\cdot))$ defines an irreducible representation of $\widehat{\mathfrak{gl}}_2'$ in $\mathcal{H}_q$. Moreover

$$\pi_q(\tau(a(t)))\Omega_q = 0$$

whenever $a(t) = \sum_{k\geq 0} a_k \otimes t^k$ with a_0 a strictly upper triangular matrix, and

$$\pi_q(\tau(e_{i,i}(0)))\Omega_q = \sum_{k\in\mathbf{Z}} E_{2k+i,2k+i}\Omega_q = (\sum_{k\in\mathbf{Z}} \lambda_{2k+i}) \cdot \Omega_q$$

where

$$\lambda_j = \lambda_{2k+i} = \begin{cases} 1 & \text{, for } 0 \leq j = 2k+i < q \\ -1 & \text{, for } q \leq j = 2k+i < 0 \\ 0 & \text{, otherwise} \end{cases}$$

As earlier is $\pi_q(\tau(c)) = 1$.

Then we are ready to discuss representations of $\widehat{\mathfrak{sl}}_2$ in details. Repeat that the loop algebra $\widetilde{\mathfrak{sl}}_2$, the Kac-Moody algebras $\widehat{\mathfrak{sl}}_2' = \widetilde{\mathfrak{sl}}_2 \oplus \mathbf{C}\cdot c$ and $\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}_2' \oplus \mathbf{C}\cdot d$ are Lie subalgebras of, and defined similar to, the corresponding Lie algebras $\widetilde{\mathfrak{gl}}_2$, $\widehat{\mathfrak{gl}}_2'$ and $\widehat{\mathfrak{gl}}_2$, respectively. Put

$$x = e_{1,2} \quad , \quad y = e_{2,1} \quad , \quad h = e_{1,1} - e_{2,2}$$

and

$$\mathfrak{n}_+ = \mathbf{C}\cdot x \quad , \quad \mathfrak{n}_- = \mathbf{C}\cdot y \quad , \quad \mathfrak{h} = \mathbf{C}\cdot h$$

The Cartan subalgebra is one-dimensional, in this case. Subsequently we write the simple Lie algebra $\mathfrak{sl}_2$ as

$$\mathfrak{sl}_2 = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

Then define

$$\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus (\mathfrak{sl}_2 \otimes t\mathbf{C}[t])$$

and

$$\hat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus \left(\mathfrak{sl}_2 \otimes t^{-1}\mathbf{C}[t^{-1}]\right)$$

where $\mathbf{C}[t]$ and $\mathbf{C}[t^{-1}]$ denote polynomials in t and t^{-1}, respectively, $t\mathbf{C}[t]$ and $t^{-1}\mathbf{C}[t^{-1}]$ those with vanishing constant term, respectively. Thus $\tau(a)$ is a strictly upper triangular matrix (in $\mathfrak{a}'_\infty$) if and only if $a \in \hat{\mathfrak{n}}_+$, where τ is the injective homomorphism of $\widetilde{\mathfrak{gl}}_2$ into $\mathfrak{a}'_\infty$, discussed earlier.

The Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{sl}}_2$ is spanned by $\hat{h}_1 = e_{1,1}(0) - e_{2,2}(0)$, c and d, however we will usually prefer the basis $\{\hat{h}_0 = c - \hat{h}_1, \hat{h}_1, d\}$ for $\hat{\mathfrak{h}}$. We may write

$$\hat{\mathfrak{sl}}_2 = \hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_-$$

The linear functionals ω_0 and ω_1 on $\mathfrak{h}$ given by

$$\omega_j(\hat{h}_i) = \delta_{i-j}$$

and

$$\omega_j(d) = 0$$

for $i, j = 0, 1$, are usual called the fundamental weights. Observe that

$$\begin{aligned}
\pi_q(\tau(\hat{h}_1))\Omega_q &= (\pi_q(\tau(e_{1,1}(0))) - \pi_q(\tau(e_{2,2}(0))))\,\Omega_q \\
&= (\sum_{k \in \mathbf{Z}} \lambda_{2k+1} - \sum_{k \in \mathbf{Z}} \lambda_{2k+2})\Omega_q \\
&= \omega_{(q \bmod 2)}(\hat{h}_1)\Omega_q \\
&= \omega_p(\hat{h}_1)\Omega_q
\end{aligned}$$

where $p = p(q) = (q \bmod 2) \in \{0, 1\}$ and

$$\begin{aligned}
\pi_q(\tau(\hat{h}_0))\Omega_q &= (\pi_q(c) - \pi_q(\tau(\hat{h}_1)))\Omega_q \\
&= (1 - \omega_p(\hat{h}_1))\Omega_q \\
&= \omega_p(\hat{h}_0)\Omega_q
\end{aligned}$$

That is

$$\pi_q(\tau(\hat{h}_i)) = \omega_p(\hat{h}_i)\Omega_q$$

where $p = q \bmod 2$. If we put $\pi_q(\tau(d))\Omega_q = 0$ (as it follows later on that we may) then is $\pi_q(\tau(\hat{h}))\Omega_q = \omega_p(\hat{h})\Omega_q$ for all $\hat{h} \in \mathfrak{h}$.

Following [K-R, p 101] we define $\lambda \in \hat{\mathfrak{h}}^*$ to be a highest weight and π_λ a highest weight representation of the Lie algbra $\hat{\mathfrak{sl}}_2$ if the representation space $\mathcal{L}(\lambda)$ contains a non-zero vector ν_λ such that π_λ is irreducible and fulfils

$$\pi_\lambda(\hat{\mathfrak{n}}_+)\nu_\lambda = 0$$

together with

$$\pi_\lambda(\hat{h})\nu_\lambda = \lambda(\hat{h})\nu_\lambda$$

for $\hat{h} \in \hat{\mathfrak{h}}$. The distinguished vector ν_λ is called the highest weight vector.

It immediately follows that for $q \in \mathbf{Z}$, the above constructed representation $\pi_q(\tau(\cdot))$ of $\widehat{\mathfrak{sl}}_2$ fulfils that $\pi_q(\tau(\hat{\mathfrak{n}}_+))\Omega_q = 0$ and $\pi_q(\tau(\hat{h}))\Omega_q = \omega_p(\hat{h})\Omega_q$, where $p = q \bmod 2$. So we are rather close to have constructed a highest weight representation of $\widehat{\mathfrak{sl}}_2$ with Ω_q as highest weight vector, but $\pi_q(\tau(\cdot))$ is not irreducible. However, each $\mathcal{H}_q$ does have a proper invariant subspace $\mathcal{H}'_q$ containing Ω_q, which does not itself allow any proper invariant subspace containing Ω_q. Thereby we will arrive with a highest weight representation of $\widehat{\mathfrak{sl}}_2$.

Recall that $\pi_q(\tau(\cdot))$ of $\widehat{\mathfrak{gl}}'_2$ in $\mathcal{H}_q$ is irreducible, since $\tau(\widehat{\mathfrak{gl}}'_2)$ contains the shift operators s_k, $k \in \mathbf{Z}$. As shown earlier is $s_{2k} \notin \tau(\widehat{\mathfrak{sl}}'_2)$ but $s_{2k+1} \in \tau(\widehat{\mathfrak{sl}}'_2)$, which leads us to define

$$\mathcal{H}'_q = \{F \in \mathcal{H}_q : \pi_q(s_{2k})F = 0, k \in \mathbf{N}\}$$

It follows that $\mathcal{H}'_q$ is invariant under $\pi_q(\tau(\widehat{\mathfrak{sl}}'_2))$, because $[s_{2k}, \tau(\widehat{\mathfrak{sl}}'_2)] = 0$, as shown earlier. Observe that

$$\pi_q(s_{-2k_1+1}) \cdot \ldots \cdot \pi_q(s_{-2k_n+1})\Omega_q$$

with $k_1 \geq \cdots \geq k_n > 0$, span a dense set in $\mathcal{H}'_q$, since

$$\begin{aligned}
&\pi_q(s_{2k})\pi_q(s_{-j_1}) \cdot \ldots \cdot \pi_q(s_{-j_n})\Omega_q \\
&\quad = \pi_q(s_{-j_1}) \cdot \ldots \cdot \pi_q(s_{-j_n})\pi_q(s_{2k})\Omega_q \\
&\qquad + \sum_{i=1}^{n} \delta_{2k-j_i}\, 2k\, \pi_q(s_{-j_1}) \cdot \ldots \cdot \pi_q(s_{-j_{i-1}})\pi_q(s_{-j_{i+1}}) \cdot \ldots \cdot \pi_q(s_{-j_n})\Omega_q
\end{aligned}$$

Thus we have the following theorem.

Theorem 31 *For each $q \in \mathbf{Z}$, π_q is a unitary highest weight representation of $\widehat{\mathfrak{sl}}_2$ with highest weight ω_p, $p = q \bmod 2$, in $\mathcal{H}'_q$, and highest weight vector Ω_q.*

Proof. An immediately consequence of the above and the fact that the inner product defines a contravariant positive definite Hermitian form with respect to the anti-linear anti-involution given by the restriction of that on $\widetilde{\mathfrak{gl}}_2$ and by $\omega(c) = c$ together with $\omega(d) = d$. ∎

Due to the fact that the highest weight, appearing in the theorem 31, are fundamental weights, we will call the corresponding highest weight representations for fundamental representations. Since there in the present case are two possible different fundamental weights, ω_0 and ω_1, it follows that there exist only two essentially different fundamental representations $\pi_0(\tau(\cdot))$ and $\pi_1(\tau(\cdot))$, because if $p = q \bmod 2$ then $\omega_{(p \bmod 2)} = \omega_{(q \bmod 2)}$ and the mapping given by $\Omega_q \to \Omega_p$ and

$$\pi_q(s_{-2k_1+1}) \cdot \ldots \cdot \pi_q(s_{-2k_n+1})\Omega_q \to \pi_p(s_{-2k_1+1}) \cdot \ldots \cdot \pi_p(s_{-2k_n+1})\Omega_p$$

extends by linearity and continuity to a unitary operator U from $\mathcal{H}'_q$ onto $\mathcal{H}'_p$, so $\pi_p(\tau(\cdot))$ and $\pi_q(\tau(\cdot))$ become unitarily equivalent, $\pi_p(\tau(\cdot)) = U\pi_q(\tau(\cdot))U^*$.

Corollary 32 *The representations $\pi_\lambda = \pi_0(\tau(\cdot))^{\otimes k_0} \otimes \pi_1(\tau(\cdot))^{\otimes k_1}$, of $\widehat{\mathfrak{sl}}_2$ in $\mathcal{L}(\lambda) = \pi_\lambda(\widehat{\mathfrak{sl}}_2)(\Omega_0^{\otimes k_0} \otimes \Omega_1^{\otimes k_1})$, where $\lambda = k_0\omega_0 + k_1\omega_1$ and $(k_0, k_1) \in (\mathbf{N} \cup \{0\})^2 \setminus \{(0,0)\}$ are unitary highest weight representations with highest weight λ and highest weight vector $\Omega_0^{\otimes k_0} \otimes \Omega_1^{\otimes k_1}$.*

Proof. As mentioned earlier, the tensor product of unitary highest weight representations (which are irreducible by definition) defines a unitary highest weight representation in the highest component, i.e. the vector space generated by the tensor product of the highest weight vectors, with highest weight given by the sum of the highest weights corresponding to the involved representations. ∎

The converse of the above corollary is included in the following proposition.

Proposition 33 *The highest weight representations π_λ of $\widehat{\mathfrak{sl}}_2$ in $\mathcal{L}(\lambda)$ is unitary if and only if $\lambda(\hat{h}_i) \in \mathbf{N}$, for $i = 0, 1$, and $\lambda(d) \in \mathbf{R}$.*

Proof. The if part follows directly from the above corollary, since we may subtract $\lambda(d)$ from d giving $d' = d - \lambda(d)$, thus $\lambda(d') = 0$.

The only if part follows since we may identify $\mathfrak{sl}_2$ naturally with the subalgebra generated by

$$x(0) = e_{1,2}(0) \quad , \quad y(0) = e_{2,1}(0) \quad , \quad h(0) = \hat{h}_1 = e_{1,1}(0) - e_{2,2}(0)$$

of $\hat{\mathfrak{sl}}_2$. The restriction of the unitary representation π_λ to $\mathfrak{sl}_2$ is still unitary. Moreover, it becomes irreducible in $\{\pi_\lambda(y(0))^k \nu_\lambda : k \in \mathbf{N} \cup \{0\}\}$, where ν_λ denote the highest weight vector, because $\pi_\lambda(x(0))\nu_\lambda = 0$ and $\pi_\lambda(h(0))\nu_\lambda = \lambda(h(0))\nu_\lambda$. Thus π_λ is an irreducible unitary highest weight representation in the above space. In what follows we will suppress the representation symbol π_λ. So the action of $\mathfrak{sl}_2$ in $\{y(0)^k\nu_\lambda : k \in \mathbf{N} \cup \{0\}\}$ is irreducible and unitary. Direct calculation using the commutation relations gives that

$$\begin{aligned} x(0)y(0)^k\nu_\lambda &= \sum_{i=0}^{k-1} y(0)^i[x(0), y(0)]y(0)^{k-i-1}\nu_\lambda + y(0)^k x(0)\nu_\lambda \\ &= \sum_{i=0}^{k-1} y(0)^i h(0) y(0)^{k-i-1}\nu_\lambda \end{aligned}$$

and

$$\begin{aligned} h(0)y(0)^n\nu_\lambda &= y(0)^n (h(0) - 2n)\nu_\lambda \\ &= y(0)^n (\lambda(h(0)) - 2n)\nu_\lambda \\ &= (\lambda(h(0)) - 2n)\, y(0)^n\nu_\lambda \end{aligned}$$

by repeatedly use of

$$\begin{aligned} h(0)y(0) &= [h(0), y(0)] + y(0)h(0) \\ &= -2y(0) + y(0)h(0) \\ &= y(0)(h(0) - 2) \end{aligned}$$

So

$$\begin{aligned} x(0)y(0)^k\nu_\lambda &= \sum_{i=0}^{k-1} y(0)^i (\lambda(h(0)) - 2(k-i-1))\, y(0)^{k-i-1}\nu_\lambda \\ &= \sum_{i=0}^{k-1} (\lambda(h(0)) - 2(k-i-1))\,(y(0)^{k-1}\nu_\lambda) \\ &= \left(k\lambda(h(0)) - 2\frac{1}{2}k(k-1)\right)(y(0)^{k-1}\nu_\lambda) \\ &= k\,(\lambda(h(0)) - k + 1)\,(y(0)^{k-1}\nu_\lambda) \end{aligned}$$

Then for $\lambda(h(0)) \notin \mathbf{N} \cup \{0\}$ is all $y(0)^k \nu_\lambda \neq 0$, by iteration, and because $y(0)^k \nu_\lambda$, $k \in \mathbf{N} \cup \{0\}$, are linearly independent, since different vectors corresponds to different eigenvalues of $h(0)$, by the above calculation, it follows that $\{y(0)_k \nu_\lambda : k \in \mathbf{N} \cup \{0\}\}$ span an infinite dimensional space. But the only unitary irreducible representations of $\mathfrak{sl}_2$ with the given involution are the finite dimensional ones. Since the compact real form is the real subspace $\{a \in \mathfrak{sl}_2 : a^* = -a\}$, which in fact equals $\mathfrak{su}(2, \mathbf{R})$, and the unitary irreducible representation restricted to $\mathfrak{su}(2, \mathbf{R})$ exponentiate to a unitary irreducible group representation of the corresponding Lie group $Su(2, \mathbf{R})$, the claim then follows, because all unitary irreducible representations of $Su(2, \mathbf{R})$ are known and are finite dimensional, because $Su(2, \mathbf{R})$ is a compact group (see [Dix, corollary 15.1.4]). Thus is $\lambda(h(0)) \in \mathbf{N} \cup \{0\}$ and then also $\lambda(\hat{h}_1) \in \mathbf{N} \cup \{0\}$, since $\hat{h}_1 = h(0)$. We emphasize that if $\lambda(h(0)) = n - 1 \in \mathbf{N} \cup \{0\}$ then the representation space become $\{y(0)^k \nu_\lambda : k = 0, 1, \ldots, n-1\}$, which has dimension $n \in \mathbf{N}$.

There is another copy of $\mathfrak{sl}_2$ in $\widehat{\mathfrak{sl}}_2$ than the one considered above, namely the Lie algebra spanned by the elements

$$x_1 = e_{2,1}(1) \quad , \quad y_1 = e_{1,2}(-1) \quad , \quad h_1 = [x_1, y_1] = c - \hat{h}_1 = \hat{h}_0$$

since

$$[h_1, x_1] = 2x_1 \qquad \text{and} \qquad [h_1, y_1] = -2y_1$$

by direct calculation using the earlier derived commutator relations (on $\widehat{\mathfrak{gl}}_2'$) $[a(k), b(l)] = [a, b]_0(k+l) + k\delta_{k+l}\mathrm{Tr}(ab) \cdot c$ and that $[x_1, y_1] = h_1$ by definition. Now, by considerations as above it follows that $\lambda(h_1) = \lambda(\hat{h}_0) \in \mathbf{N} \cup \{0\}$. Then is $\lambda(c) = \lambda(h_1) + \lambda(h(0)) \in \mathbf{N} \cup \{0\}$, since $\hat{h}_1 = h(0)$. Hereby the proposition is proved. ∎

Since unitarity of an irreducible representation of $\widehat{\mathfrak{sl}}_2$ gives that $\lambda(c) \in \mathbf{N} \cup \{0\}$ and that this distinguished value somehow characterizes the representation (see below) it is sometimes referred to as *the level* of the particular representation.

We will now demonstrate how the Goddard-Kent-Olive (GKO) construction gives a recipe to construct a series of unitary representations of the Virasoro algebra with central charge running through a discrete subset of $[0, 1[$. The GKO construction uses the Sugawara construction, described in the last part of section 3.1, together with theorem 26 of section 3.2. To obtain the above mentioned serie of representations, the GKO construction will be build on $\widehat{\mathfrak{sl}}_2'$ and we therefore limit ourself to this case in the following considerations. However, the GKO construction does have meaning, when it is build on an arbitrary simple Lie algebra.

Put $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and consider two representations of $\widehat{\mathfrak{sl}}_2$ in vector spaces $\mathcal{L}(\lambda) = \mathcal{H}'_q$ for some $q \in \mathbf{Z}$ and $\mathcal{L}(\mu) = \mathcal{H}'_p$ for some $p \in \mathbf{Z}$ with levels $m_\lambda = \lambda(c)$ and $m_\mu = \mu(c)$, respectively. In what follows we will suppress the representation symbols, as we have done before. Then the action of $\hat{\mathfrak{g}}' = \widehat{\mathfrak{sl}}_2' \oplus \widehat{\mathfrak{sl}}_2'$ in $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$ is given by

$$(x(n) \oplus y(m))(v \otimes w) = (x(n)(v)) \otimes w + v \otimes (y(m)(w))$$

for any $v \otimes w \in \mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$ and $n, m \in \mathbf{Z}$. Then the Sugawara construction (see section 3.1) of the representations $x(n) \otimes 1$ and $1 \otimes y(m)$ provides us with Virasoro operators $L_k^{(\lambda)} \otimes 1$ and $1 \otimes L_k^{(\mu)}$, respectively, hence

$$\begin{aligned} L_k^{\mathfrak{g}} &= L_k^{(\lambda)} \otimes 1 + 1 \otimes L_k^{(\mu)} \\ &= \frac{1}{2(Q+m_\lambda)} \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} (:x_i(l) x_i(k-l):) \otimes 1 \\ &\quad + \frac{1}{2(Q+m_\mu)} \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} 1 \otimes (:y_i(l) y_i(k-l):) \end{aligned}$$

defines a representation of the Virasoro algebra in $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$, with central charge

$$c_{\mathfrak{g}} = c_\lambda + c_\mu = 3\left(\frac{m_\lambda}{Q+m_\lambda} + \frac{m_\mu}{Q+m_\mu}\right)$$

since

$$\begin{aligned} [L_n^{\mathfrak{g}}, L_k^{\mathfrak{g}}] &= (n-k) L_{n+k}^{(\lambda)} \otimes 1 + \delta_{n+k} \frac{n^3-n}{12} \cdot \frac{m_\lambda \dim(\mathfrak{sl}_2)}{Q+m_\lambda} \\ &\quad + (n-k) 1 \otimes L_{n+k}^{(\mu)} + \delta_{n+k} \frac{n^3-n}{12} \cdot \frac{m_\mu \dim(\mathfrak{sl}_2)}{Q+m_\mu} \\ &= (n-k) L_{n+k}^{\mathfrak{g}} + \delta_{n+k} \frac{n^3-n}{12} \cdot c_{\mathfrak{g}} \end{aligned}$$

and the dimension of $\mathfrak{sl}_2$ is 3. However, there is also a level $m_\lambda + m_\mu$ representation of $\hat{\mathfrak{sl}}'_2$ ($\cong \{x \otimes y \in \hat{\mathfrak{sl}}'_2 \otimes \hat{\mathfrak{sl}}'_2 : x = y\}$) in $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$, namely the restriction of the former to the diagonal (or $x(n) \to x(n) \otimes 1 + 1 \otimes x(n)$). By the Sugawara construction of this representation, we arrive with the following Virasoro operators

$$\begin{aligned} L'_k &= \frac{1}{2(Q + m_\lambda + m_\mu)} \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} :(x_i(l) \otimes 1 + 1 \otimes x_i(l)) \\ &\qquad \cdot (x_i(k-l) \otimes 1 + 1 \otimes x_i(k-l)): \\ &= \frac{1}{2(Q + m_\lambda + m_\mu)} \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} ((:x_i(l) x_i(k-l):) \otimes 1 \\ &\qquad + 1 \otimes (:x_i(l) x_i(k-l):) + 2 x_i(l) \otimes x_i(k-l)) \end{aligned}$$

with central charge $c' = 3\frac{m_\lambda + m_\mu}{Q + m_\lambda + m_\mu}$, since $\dim(\mathfrak{sl}_2) = 3$. Thus theorem 26 gives us Virasoro operators

$$\begin{aligned} L_k &= L_k^{\mathfrak{g}} - L'_k \\ &= \left(\frac{1}{2(Q + m_\lambda)} - \frac{1}{2(Q + m_\lambda + m_\mu)}\right) \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} (:x_i(l) x_i(k-l):) \otimes 1 \\ &\quad + \left(\frac{1}{2(Q + m_\mu)} - \frac{1}{2(Q + m_\lambda + m_\mu)}\right) \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} 1 \otimes (:x_i(l) x_i(k-l):) \\ &\quad - \frac{1}{Q + m_\lambda + m_\mu} \sum_{l \in \mathbf{Z}} \sum_{i=1}^{3} x_i(l) \otimes x_i(k-l) \end{aligned}$$

Hereby we get a representation of the Virasoro algebra, in $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$, with central charge

$$c = c_{\mathfrak{g}} - c' = 3\left(\frac{m_\lambda}{Q + m_\lambda} + \frac{m_\mu}{Q + m_\mu} - \frac{m_\lambda + m_\mu}{Q + m_\lambda + m_\mu}\right)$$

One can calculate L_0 explicetely. Notice that $\Omega_\lambda = 2(Q + m_\lambda)d + 2T(0)$ commutes with every elements of $\hat{\mathfrak{sl}}_2$, where

$$T(0) = \frac{1}{2} \sum_{i=1}^{3} x_i(0) x_i(0) + \sum_{l \in \mathbf{N}} \sum_{i=1}^{3} x_i(-l) x_i(l)$$

see section 3.1, and Ω_λ is the Casimir operator of $\widehat{\mathfrak{sl}}_2$ in $\mathcal{L}(\lambda)$ since

$$[d, \Omega_\lambda] = 0 + 2[d, T(0)] = 0$$

and

$$[x(n), \Omega_\lambda] = -2(Q + m_\lambda)\, n\, x(n) + 2(Q + m_\lambda)\, n\, x(n) = 0$$

by direct calculation using the various commutation relations on $\widehat{\mathfrak{sl}}_2$ given in section 3.1 together with lemma 24. Trivially $[c_\lambda, \Omega_\lambda] = 0$. We have the similar relations with μ instead of λ and for the diagonal representation too. Then

$$L_0^{(\lambda)} = \frac{1}{Q + m_\lambda} T(0) = \frac{\Omega_\lambda}{2(Q + m_\lambda)} - d,$$

$$L_0^{(\mu)} = \frac{\Omega_\mu}{2(Q + m_\mu)} - d$$

and

$$L_0' = \frac{\Omega'}{2(Q + m_\lambda + m_\mu)} - d \otimes 1 - 1 \otimes d$$

where Ω' is the Casimir of $\widehat{\mathfrak{sl}}_2$ in $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)$, whenever we are considering admissible representations, as the highest weight representations indeed are. We therefore get the following expression for L_0

$$\begin{aligned} L_0 &= \left(\frac{\Omega_\lambda}{2(Q + m_\lambda)} - d\right) \otimes 1 + 1 \otimes \left(\frac{\Omega_\mu}{2(Q + m_\mu)} - d\right) \\ &\quad - \left(\frac{\Omega'}{2(Q + m_\lambda + m_\mu)} - d \otimes 1 - 1 \otimes d\right) \\ &= \frac{1}{2}\left(\frac{(\lambda, \lambda + 2\rho)}{Q + m_\lambda} + \frac{(\mu, \mu + 2\rho)}{Q + m_\mu} - \frac{\Omega'}{Q + m_\lambda + m_\mu}\right) \end{aligned}$$

where we have used that the eigenvalue of Ω_ν is $(\nu, \nu + 2\rho)$, for $\nu = \lambda, \mu$, and ρ is the sum of fundamental weights (see [Hu, section 22.3] or [K-R, section 10.1-10.2]). However, we will not really need the expression for L_0, but rather focus on the possible value of c. We therefore return to the Cartan subalgebra $\hat{\mathfrak{h}}$ of $\widehat{\mathfrak{sl}}_2$. Choose $\{b = \hat{h}_1 = e_{1,1}(0) - e_{2,2}(0), c, d\}$ as a basis for $\hat{\mathfrak{h}}$, hence $\hat{\mathfrak{h}} = \mathbf{C} \cdot b \oplus \mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$. The bilinear form on $\widehat{\mathfrak{gl}}_2$ (from proposition 20, with $s = 0$ and $(a(t), b(t))_0 = \mathrm{Tr}(a(t)b(t))$, $a(t), b(t) \in \widehat{\mathfrak{gl}}_2$) restricts to a bilinear form on $\widehat{\mathfrak{sl}}_2$ and become non-degenerated when restricted further to $\hat{\mathfrak{h}}$. In fact is

$$\begin{aligned}(b,b) &= \mathrm{Tr}\left((e_{1,1}(0)-e_{2,2}(0))(e_{1,1}(0)-e_{2,2}(0))\right)\\ &= \mathrm{Tr}\left(e_{1,1}(0)+e_{2,2}(0)\right)=2\end{aligned}$$

and

$$(c,d)=(d,c)=1$$

all other evaluations of the form do vanish. Then we may identify $\hat{\mathfrak{h}}$ with the dual space $\hat{\mathfrak{h}}^*$ via this form. In the highest weight representation $\mathcal{L}(\lambda)$ of $\widehat{\mathfrak{sl}}_2$ the action of $\hat{\mathfrak{h}}$ on the highest weight vector ν_λ is given as

$$\hat{h}(\nu_\lambda)=\lambda(\hat{h})\nu_\lambda=(\lambda,\hat{h})\nu_\lambda$$

for $\hat{h}\in\hat{\mathfrak{h}}$. With the above choice of basis the fundamental weights become $\omega_0=(d,\cdot)\cong d$ and $\omega_1=(\frac{1}{2}b,\cdot)\cong\frac{1}{2}b$, since $\omega_i(\hat{h}_j)=\delta_{i-j}$, $i=0,1$, $\hat{h}_0=c$ and $\hat{h}_1=b$. As earlier does ρ denote the sum of the fundamental weights, $\rho=\omega_0+\omega_1\cong d+\frac{1}{2}b$.

We know from proposition 33 that the representation in $\mathcal{L}(\lambda)$ is unitary if and only if λ takes the form

$$\lambda=m\cdot d+\frac{1}{2}n\cdot b+r\cdot c$$

where $r=\lambda(d)\in\mathbf{R}$, $m=\lambda(c)\in\mathbf{N}\cup\{0\}$ and $n=\lambda(b)\in\mathbf{N}\cup\{0\}$. So $c(\nu_\lambda)=\lambda(c)\nu_\lambda=m\nu_\lambda$ and $\mathcal{L}(\lambda)$ becomes a level m representation ($m_\lambda=m$). Choosing the standard basis $\{x,y,h\}$ for $\mathfrak{sl}_2$, it follows directly that $x'=y$, $y'=x$ and $h'=\frac{1}{2}h$ defines the dual basis with respect to the trace form, $(a,b)_0=\mathrm{Tr}(ab)$, restricted from $\mathfrak{gl}_2$.

By use of the commutation relations on $\mathfrak{sl}_2$ it follows that the Casimir operator becomes

$$\Omega_0=xx'+hh'+yy'=xy+\frac{1}{2}h^2+yx=\frac{1}{2}h^2+h+2yx$$

and in the adjoint representation

$$\Omega_0^{(\mathrm{ad})}=(\mathrm{ad}x)(\mathrm{ad}y)+\frac{1}{2}(\mathrm{ad}h)(\mathrm{ad}h)+(\mathrm{ad}y)(\mathrm{ad}x)$$

which has eigenvalue $2Q = 4$, since it is known that it acts as a scalar $2Q$ and it follows directly that $(\Omega_0^{(\mathrm{ad})})(h) = 4h$. So $Q = 2$. We now choose some particular weights $\mu = d$ (corresponding to $m = 1$, $n = 0$ and $r = 0$) and $\lambda = md + \frac{1}{2}nb$ (corresponding to $r = 0$) with $m, n \in \mathbf{N} \cup \{0\}$. Thus the previously introduced GKO construction for $\mathfrak{sl}_2$ provide us with unitary representations of the Virasoro algebra, with $Q = 2$ and $m_\mu = \mu(c) = (d, c) = 1$. It becomes a highest weight representation in the highest component. The operator L_0, dependent of m and n, becomes

$$\begin{aligned} L_0 &= \frac{1}{2}\left(\frac{(md + \frac{1}{2}nb, md + \frac{1}{2}nb + 2d + b)}{2+m} + \frac{(d, d + 2d + b)}{2+1} - \frac{\Omega'}{2+m+1}\right) \\ &= \frac{\frac{1}{2}n^2 + n}{4(m+2)} + 0 - \frac{\Omega'}{2(m+3)} = \frac{n(n+2)}{4(m+2)} - \frac{\Omega'}{2(m+3)} \end{aligned}$$

where $m, n \in \mathbf{N} \cup \{0\}$. The interesting central charge in this particular case becomes

$$c_m = 3\left(\frac{m}{2+m} + \frac{1}{2+1} - \frac{m+1}{2+m+1}\right) = 1 - \frac{6}{(m+2)(m+3)}$$

where $m \in \mathbf{N} \cup \{0\}$.

Hereby we have succeded in constructing a series of unitary highest weight representations of the Virasoro algebra in the highest component of $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) = \mathcal{H}'_q \otimes \mathcal{H}'_p$ generated by $\nu_\lambda \otimes \nu_\mu (= \Omega_q \otimes \Omega_p)$, with central charge given by the above discrete series $c_m \in [0, 1[$. We emphasize that for $m = 1$ is $c_1 = \frac{1}{2}$.

It shall be pointed out that the eigenvalue $h \in \mathbf{R}$ of L_0 also belongs to a discrete series (here h doesn't denote an element of $\mathfrak{h}$ or $\hat{\mathfrak{h}}$). In fact, for a given $m \in \mathbf{N} \cup \{0\}$, or equivalent a given c_m, is the $\frac{1}{2}(m+1)(m+2)$ numbers

$$h_m(r, s) = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}$$

where $r, s \in \mathbf{N}$ such that $1 \le s \le r \le m+1$, the only possible value h can take. We will not dwell at the proof for this result, but mention that it is based on the celebrated Kac determinant formula (see for example [K-R, chapter 8-12]). From this formula it follows that for $m = 1$, or equivalently

$c = \frac{1}{2}$, is the only possible value of h giving unitary representations those where

$$h = h_1(1,1) = 0 \quad , \quad h = h_1(1,2) = \frac{1}{2} \quad , \quad h = h_1(2,2) = \frac{1}{16}$$

a result which we have used earlier in section 3.2.

Chapter 4

Applications

4.1 The Loop group LS^1.

In this section we consider the particular loop group LS^1 viewed as an abelian subgroup of the restricted unitary group, discussed in section 1.5. We will, in particular, use the infinite dimensional spin representation, treated in detail in sections 1.4 and 1.5, in our discussion of the particular loop group LS^1, also known as the loop circle. Section 3.1 on loop algebras will serve as basis knowledge.

The loop groups has been studied intensively in the, already fameous, book by Pressley and Segal [P-S] from 1986. However, their approach is slightly different form ours. Anyway [P-S] will serve as a foundation of this section. The loop circle has also been considered by Lundberg in [Lu 2], we will follow this approach. Severel others have studied the loop groups (see [P-S] for further references), but we will only add [Mi] to the list, though.

The applications of loop groups is various, for example do loop groups appear in two dimensional quantum field theory and more recently they have been put to extensive use in connection with the socalled completely integrable systems of partial differential equations (see [P-S] and [Mi] for further details).

Consider the complex Hilbert space $\mathcal{H} = L_2(S^1)$ of squared-integrable functions on the unit circle S^1. The inner product on $\mathcal{H}$ is given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(\theta)} g(\theta)\, d\theta$$

for any $f, g \in \mathcal{H}$.

We may choose $\{e_k \in \mathcal{H} : k \in \mathbf{Z}\}, e_k(\theta) = e^{ik\theta}$, as an orthonormal basis for $\mathcal{H}$. We then have a canonical splitting, or polarization, of $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = \text{span}\{e_k : k \geq 0\}$ and $\mathcal{H}_- = \text{span}\{e_k : k < 0\}$. We denote the projection onto $\mathcal{H}_-$ by P.

The loop group LG is the group of parametrized loops in the group G, i.e. LG is the group of smooth maps (loops) from the circle S^1 into G. In the present case we only consider $G = S^1$. Hence LS^1 consist of smooth endomorphisms of S^1. Then any element in LS^1 can be written in the form e^{iF}, where F is a smooth function from S^1 into $\mathbf{R}$ such that $F(\theta + 2\pi) = F(\theta) + 2\pi \cdot n_F$, for some integer n_F, which is the winding number of e^{iF} (see for example [P-S, p. 59]). Notice that $e^{iF}: S^1 \to S^1$, $e^{iF(\theta+2\pi)} = e^{iF(\theta)}$ and e^{iF} is certainly smooth.

Due to [P-S, p. 79] we may realize the *loop group* LS^1 as multiplication operators of the form $M(e^{iF})$ on $\mathcal{H}$, i.e. $(M(e^{iF})g)(\theta) = e^{iF(\theta)} \cdot g(\theta)$, for any $g \in \mathcal{H}$. In the following we simply write e^{iF} for $M(e^{iF})$, whereby we identify the multiplication operator $M(e^{iF})$ on $\mathcal{H}$ with the group element $e^{iF} \in LS^1$. In the present case it appears that it is sufficient to demand that the real valued function F on S^1 is C^1, i.e. $F \in C^1(S^1)$, and not necessarily smooth. If we put $f_F(\theta) = F(\theta) - n_F \cdot \theta$ then $f_F(\theta + 2\pi) = f_F(\theta)$. Hence $f_F \in C^1(S^1)$ is real and invariant with respect to the shift $\theta \to \theta + 2\pi$. Therefore f_F can be expanded as a real Fourier series $f_F = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}} f_k \cdot e_k$, where the Fourier components f_k are given by $f_k = \langle e_k, f_F\rangle = \frac{1}{2\pi}\int_0^{2\pi} f_F(\theta)e^{-ik\theta}\,d\theta$. Separating out the zero-Fourier component $f_0 = \frac{1}{2\pi}\int_0^{2\pi} f_F(\theta)\,d\theta \in \mathbf{R}$ we may write $f_F = f_0 + \frac{1}{2\pi}\sum_{k\in\mathbf{Z}\setminus\{0\}} f_k \cdot e_k$, where $e_k(\theta) = e^{ik\theta}$ and $\overline{f}_{-k} = f_k$ (since f_F is real). Put $f(\theta) = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}\setminus\{0\}} f_k \cdot e_k(\theta) = f_F(\theta) - f_0$, and notice that the mean value of f is $\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = 0$. We finally come up with the following splitting of F

$$F(\theta) = n_F \cdot \theta + f_0 + f(\theta)$$

where $n_F \in \mathbf{Z}$ is the winding number of e^{iF}, $f_0 \in \mathbf{R}$ is the mean value of $F(\theta) - n_F \cdot \theta$ and $f \in C^1(S^1)$ is real and has zero-trace, $\int_0^{2\pi} f(\theta)\,d\theta = 0$. If we put $F_0(\theta) = n_F \cdot \theta + f_0$, such that $F = F_0 + f$, then $e^{iF} \in LS^1$ can be factorized as $e^{iF_0} \cdot e^{if}$. The subgroup of LS^1 consisting of elements of the form e^{if} will be called *the special loop group*, denoted by SLS^1, and the subgroup of LS^1 generated by elements of the form e^{iF_0} will be called *the charge subgroup* of LS^1, denoted by C. Thus LS^1 may be considered as the product group $SLS^1 \times C$.

We consider the case of the special loop group SLS^1 first. Below, we will show that SLS^1 may be considered as a subgroup of the restricted unitary group $\mathcal{U}_2(\mathcal{H}, P)$, studied in section 1.5, where P is the projection onto $\mathcal{H}_-$. Furthermore, we study the explicit spin representation, which is a projective representation of positive energy of SLS^1.

According to the results in section 1.5, we only have to verify that $if \in u_2(\mathcal{H}, P)$, whence $e^{if} \in \mathcal{U}_2(\mathcal{H}, P)$. Evidently $(if)^* = -if$. The Hilbert-Schmidt stipulation of $[P, if]$ is equivalent to that of $Pf(1-P) = P[P, f]$, since $[P, f] = Pf(1-P) - (1-P)fP$ and $Pf(1-P)$ is Hilbert-Schmidt if and only if $(1-P)fP = (Pf(1-P))^*$ is. Therefore it is sufficient to prove that $Pf(1-P)$ is Hilbert-Schmidt. Let χ denote the indicator function of $\mathbf{N} \cup \{0\}$, i.e. $\chi(k) = 1$ for $k \in \mathbf{N} \cup \{0\}$ and zero otherwise. Then

$$\langle e_k, Pf(I-P)fPe_k\rangle = \sum_{n\in\mathbf{Z}} \chi(-k-1)\chi(n+k) \cdot |f_n|^2$$

by direct calculation using the Fourier series for f (and the fact that $\overline{f}_{-n} = f_n$, since f is real). Hence

$$\begin{aligned}
\operatorname{Tr}(Pf(I-P)fP) &= \sum_{k\in\mathbf{Z}} \langle e_k, Pf(I-P)fPe_k\rangle \\
&= \sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \chi(n+k)\chi(-k-1) \cdot |f_n|^2 \\
&= \sum_{k\in\mathbf{N}} \sum_{n\in\mathbf{Z}} \chi(n-k) \cdot |f_n|^2 \\
&= \sum_{n\in\mathbf{N}} n \cdot |f_n|^2
\end{aligned}$$

where we have used Fubinis theorem to interchange the summation since

$$\begin{aligned}
\sum_{n\in\mathbf{N}} n \cdot |f_n|^2 &\leq \sum_{n\in\mathbf{Z}} n^2 \cdot |f_n|^2 \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f'(\theta)|^2 \, d\theta \\
&< \infty
\end{aligned}$$

due to the fact that $f \in C^1(S^1)$. We then conclude that $Pf(1-P)$ and then $[P, if]$ is Hilbert-Schmidt. So $if \in u_2(\mathcal{H}, P)$, whence $e^{if} \in \mathcal{U}_2(\mathcal{H}, P)$. Evidently SLS^1 can be realized as a subgroup of $\mathcal{U}_2(\mathcal{H}, P)$ (for an alternative proof see [P-S, p. 82-83]). Notice that we may associate the Lie algebra slS^1, consisting of skew-selfadjoint multiplication operators, if, such that $[P, if]$ is a Hilbert-Schmidt operator, with the special loop group SLS^1. Hence the *special loop algebra* slS^1 generate SLS^1 through the exponential mapping.

Due to the above inclusion of slS^1 into $u_2(\mathcal{H}, P)$ and SLS^1 into $\mathcal{U}_2(\mathcal{H}, P)$, the spin representation, constructed in sections 1.4 and 1.5, is well defined and gives a projective representation of SLS^1. We will now show that this spin representation is, in fact, of positive energy. In order to achieve this, we have to construct a Hamilton operator (to apply some meaning to the statement).

Let d_0 denote the generator for rotations in $\mathcal{H} = L_2(S^1)$, i.e. $d_0 = \frac{d}{d\theta}$ on its maximal domain $\mathcal{D}(d_0) = \{f \in \mathcal{H} : d_0 f \in \mathcal{H}\}$. It follows, by partial integration that d_0 is skew-selfadjoint. Even though d_0 is unbounded, its commutator with P still makes sense. Since $[P, d_0]e_k = Pd_0e_k - d_0Pe_k = 0$, $[P, d_0]$ does vanish on $\mathcal{D}(d_0)$, hence it makes sense and vanishes on all of $\mathcal{H}$ (as long as we do not consider other unbounded operators, we won't get into trouble). We define a Hamilton operator H on $\mathcal{F}_\wedge(\mathcal{H})$ by $H = -idU_P(d_0)$ with the dense domain $\mathcal{D}(H)$ given below. Note that $[P, d_0] = 0$ and $[\Gamma, d_0]_+ = 0$, where Γ is a given involution on $\mathcal{H}$ commuting with P (see sections 1.5 and 1.2). It follows that

$$
\begin{aligned}
H &= -idU_P(d_0) = -idU((d_0)_P) \\
&= -idU(d_0(I - 2P))
\end{aligned}
$$

where $dU((d_0)_P)$ is defined by the Fock-Cook quantization mapping (see section 1.3), since the skew-selfadjoint unbounded operator d_0 is linear. Notice that $(d_0)_P e_k = d_0(I - 2P)e_k = i\,|k| \cdot e_k$, for all $k \in \mathbf{Z}$, due to the polarization operator $J = I - 2P$ (which have eigenvalues ± 1 and corresponding eigenspaces $\mathcal{H}_\pm$). Hence on arbitrary basis product vectors $e_{k_1} \wedge \cdots \wedge e_{k_n}$ in $\wedge^n \mathcal{H}$, $n \in \mathbf{N}$, is

$$
\begin{aligned}
H(e_{k_1} \wedge \cdots \wedge e_{k_n}) &= \sum_{j=1}^{n} e_{k_1} \wedge \cdots \wedge e_{k_{j-1}} \wedge -i(d_0)_P e_{k_j} \wedge e_{k_{j+1}} \wedge \cdots \wedge e_{k_n} \\
&= \left(\sum_{j=1}^{n} |k_j| \right) \cdot (e_{k_1} \wedge \cdots \wedge e_{k_n})
\end{aligned}
$$

Moreover, $H\Omega = 0$, so the eigenspace corresponding to the eigenvalue 0 is spanned by Ω and e_0, and the eigenspace corresponding to any $m \in \mathbf{N}$ is spanned by basis product vectors $e_{k_1} \wedge \cdots \wedge e_{k_n}$ with $\sum_{j=1}^{n} |k_j| = m$. Observe that these eigenspaces are of finite dimension (each $k_j \in \{-m, -m+1, \ldots, m-1, m\}$ and $n \leq m$). The spectrum of H is $\sigma(H) = \mathbf{N} \cup \{0\}$ and evidently $H \geq 0$ (so the representation is indeed of positive energy). Notice that H indeed is unbounded and that the maximal domain

$$
\mathcal{D}(H) = \{F \in \mathcal{F}_\wedge(\mathcal{H}) : HF \in \mathcal{F}_\wedge(\mathcal{H})\}
$$

for H consists of particles (vectors) of finite energy. $\mathcal{D}(H)$ is evidently dense in $\mathcal{F}_\wedge(\mathcal{H})$, since it includes the algebraic direct sum of the energy eigenspaces and $\mathcal{F}_\wedge(\mathcal{H})$ is the (Hilbert space) direct sum of the energy eigenspaces, i.e. $\mathcal{F}_\wedge(\mathcal{H})$ is the completion of $\mathcal{D}(H)$. Since $f \in C^1(S^1)$, $\|if\|_\infty < \infty$ and the multiplication operator $M(if)$ which multiply with if is bounded. Then we define $\phi(f)$ to be the selfadjoint closure of the essential selfadjoint generator $-idU_P(if)$, for $if \in slS^1 \subset u_2(\mathcal{H}, P)$. From the commutation relations (1.17) we get

$$\begin{aligned}
[\phi(f), \phi(g)] &= -[dU_P(if), dU_P(ig)] \\
&= -dU([(if)_P, (ig)_P]) - \omega((if)_P, (ig)_P) \cdot I \\
&= -\omega_P(if, ig) \cdot I
\end{aligned}$$

on $\mathcal{D}$ since $[(if)_P, (ig)_P] = [if, ig]_P = 0$ and where the Lie algebra cocycle $\omega_P(if, ig) = \omega((if)_P, (ig)_P)$ is given by (1.16) as

$$\begin{aligned}
\omega_P(if, ig) &= \omega((if)_P, (ig)_P) \\
&= \mathrm{Tr}(Pg(I-P)fP) - \mathrm{Tr}(Pf(I-P)gP)
\end{aligned}$$

for all $if, ig \in slS^1$. This means that we have constructed a representation of the CCR-algebra (see (2.9) and definition 13 of chapter 2) in the antisymmetric Fock Hilbert space $\mathcal{F}_\wedge(\mathcal{H})$, indicating the socalled boson-fermion correspondance. We return to the boson-fermion correspondance in section 4.4.

From section 1.5 it follows that the charge operator $Q = dU(I - 2P)$ commutes with every $dU_P(A)$, $A \in u_2(\mathcal{H}, P)$. This holds especially for $A = if \in slS^1$, i.e. $[i\phi(f), Q] = 0$ on $\mathcal{D}$. Hence $W(f) = e^{i\phi(f)}$ commutes with the charge operator Q on $\mathcal{D}$, and the (corresponding) representation of the CCR-algebra is highly reducible, since each charge sector $\mathcal{H}_q$ (see section 1.5) is invariant.

Now we complexify the mapping $f \mapsto \phi(f)$ by putting $\phi(f) = \phi(f_r) + i\phi(f_i)$, for $f = f_r + if_i$ in $C^1(S^1)_\mathbb{C} = C^1(S^1) + iC^1(S^1)$ such that both f_r and f_i are real and have vanishing trace. Then this extended $\phi(f)$ is no longer selfadjoint, however, $f \mapsto \phi(f)$ is a "$*$-quantization mapping", i.e. $\phi(f)^* = \phi(\overline{f})$, on $\mathcal{D}$, since $\langle \phi^*(f)F_n, G_m\rangle = \langle F_n, (\phi(f_r) + i\phi(f_i))G_m\rangle = \langle(\phi(f_r) - i\phi(f_i)F_n, G_m\rangle = \left\langle \phi(\overline{f})F_n, G_m\right\rangle$, for any $F_n \in \wedge^n\mathcal{H}$ and $G_m \in \wedge^m\mathcal{H}$. Moreover, each $f \in C^1(S^1)_\mathbb{C}$ can be uniquely decomposed as $f = f_+ \oplus f_-$ where $f_+ = \sum_{k\in\mathbb{N}} \langle e_k, f\rangle e_k \in \mathcal{H}_+$ and $f_- = \sum_{k\in-\mathbb{N}} \langle e_k, f\rangle e_k \in \mathcal{H}_-$. Recall that $f_0 = 0$ due to the vanishing trace, notice also that $\langle f_+, f_-\rangle = 0$. Then

$\phi(f) = \phi(f_+) + \phi(f_-)$. The observation

$$\begin{aligned}\overline{(f_-)} &= \overline{\sum_{k\in -\mathbf{N}} \langle e_k, f\rangle\, e_k} = \sum_{-k\in\mathbf{N}} \left\langle e_{-k}, \overline{f}\right\rangle e_{-k} \\ &= \sum_{k\in\mathbf{N}} \left\langle e_k, \overline{f}\right\rangle e_k = (\overline{f})_+\end{aligned}$$

implies that $\overline{(f_-)} = (\overline{f})_+ = f_+$ in case f is real (notice that f_+ and f_- may be non-real even though f itself is real). Hence $\phi(f_-)^* = \phi(f_+)$, for $if \in slS^1$, and $\|\phi(f_-)\Omega\|^2 = \langle\Omega, \phi(f_+)\phi(f_-)\Omega\rangle = \mathrm{Tr}(Pf_+(I-P)f_-P) = \sum_{m>0}\sum_{n<0}\sum_{k\in\mathbf{Z}} f_m f_n \chi(-k-1)\chi(k+n)\delta_{m+n} = \sum_{n<0}\sum_{k<0}|f_n|^2\cdot\chi(k+n) = 0$, then $\phi(f_-)\Omega = 0$, for $if \in slS^1$.

We will now compute the cocycle $\omega_P(if, ig)$ explicitely. With the earlier notation we have $Pe_k = \chi(-k-1)e_k$, so

$$gPe_k = \sum_{n\in\mathbf{Z}} g_n \chi(-k-1)e_{n+k}$$

and

$$f(I-P)gPe_k = \sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} f_m g_n \chi(-k-1)\chi(k+n)e_{k+n+m}$$

Hence

$$\begin{aligned}&\mathrm{Tr}(Pf(I-P)gP) \\ &= \sum_{k\in\mathbf{Z}} \langle Pe_k, f(I-P)gPe_k\rangle \\ &= \sum_{k\in\mathbf{Z}}\sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} f_m g_n \chi(-k-1)\chi(k+n)\cdot\delta_{m+n} \\ &= \sum_{k\in-\mathbf{N}}\sum_{m\in\mathbf{Z}} f_m g_{-m}\chi(k-m) \\ &= \sum_{k\in-\mathbf{N}}\sum_{n\in\mathbf{Z}} \overline{f_n} g_n \chi(k+n) \\ &= \sum_{n\in\mathbf{N}} n\overline{f_n} g_n\end{aligned}$$

where we have used Fubini's theorem to interchange the summations, since

$$n\cdot\left|\overline{f_n}\right|\cdot|g_n| \le \frac{1}{2}\left(n^2\cdot\left|\overline{f_n}\right|^2 + |g_n|^2\right)$$

for $n \in \mathbf{N}$, and

$$\begin{aligned}\sum_{n\in\mathbf{N}} n \cdot \left|\overline{f_n}\right| \cdot |g_n| &\leq \frac{1}{2}\sum_{n\in\mathbf{N}} n^2 \cdot \left|\overline{f_n}\right|^2 + \frac{1}{2}\sum_{n\in\mathbf{N}} |g_n|^2 \\ &= \frac{1}{4\pi}\int_0^{2\pi} |f'(\theta)|^2\, d\theta + \frac{1}{4\pi}\int_0^{2\pi} |g(\theta)|^2\, d\theta\end{aligned}$$

which is finite, because $f, g \in C^1(S^1)$. Hence

$$\omega_P(if, ig) = \sum_{n\in\mathbf{N}} n \cdot \left(f_n \cdot \overline{g_n} - \overline{f_n} \cdot g_n\right)$$

for $if, ig \in slS^1$, where $f_n = \langle e_n, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$. Furthermore

$$\omega_P(if, ig) = \frac{1}{2\pi i}\int_0^{2\pi} f'(\theta)g(\theta)d\theta$$

for $if, ig \in slS^1$, since

$$\begin{aligned}\frac{1}{2\pi i}\int_0^{2\pi} f'(\theta)g(\theta)d\theta &= \sum_{n\in\mathbf{Z}} n \cdot f_n \cdot \overline{g_n} \\ &= \sum_{n\in\mathbf{N}} n \cdot \left(f_n \cdot \overline{g_n} - \overline{f_n} \cdot g_n\right)\end{aligned}$$

because $f_0 = g_0 = 0$, $f'(\theta) = \sum_{n\in\mathbf{Z}} inf_ne_n(\theta)$ and $nf_n\overline{g_n} = n\overline{f_{-n}g_{-n}} = -(-n)\overline{f_{-n}}g_{-n} = -k\overline{f_k}g_k$, for $k = -n$ and n negative. Notice that $\omega_P(\cdot,\cdot)$ evidently defines a non-degenerated symplectic form on $slS^1 \times slS^1$, since

$$\begin{aligned}\omega_P(if, ig) + \omega_P(ig, if) &= \frac{1}{2\pi i}\int_0^{2\pi} (f(\theta)g(\theta))'d\theta \\ &= \frac{1}{2\pi i}\left[f(\theta)g(\theta)\right]_0^{2\pi} \\ &= 0\end{aligned}$$

giving the anti-symmetry of the form, and $\int_0^{2\pi} f'(\theta)g(\theta)d\theta = 0$, for all $ig \in slS^1$, implies that $f'(\theta) = 0$ so $f(\theta) = f_0 = 0$, giving the non-degeneracy of the form.

The socalled two-point function becomes

$$\begin{aligned}\langle\Omega, \phi(f)\phi(g)\Omega\rangle &= -\langle\Omega, dU_P(if)dU_P(ig)\Omega\rangle \\ &= -\mathrm{Tr}(Pif(I-P)igP) \\ &= \sum_{n\in\mathbf{N}} n \cdot \overline{f_n}g_n\end{aligned}$$

by use of formula (1.15) derived in the end of section 1.5 and the above trace formula.

The functions in $i \cdot slS^1$, considered as real functions on S^1 whose integral wanish, are completely determined by their positive Fourier compenents. In fact, the Fourier transformation $F : f \to (f_k)_{k \in \mathbf{N}}$ is an isomorphism. On the space of all these Fourier transformed $(f_k)_{k \in \mathbf{N}}$, $f \in i \cdot slS^1$ is

$$\langle (f_k)_{k \in \mathbf{N}}, (g_n)_{n \in \mathbf{N}} \rangle = \sum_{k \in \mathbf{N}} k \cdot \overline{f_k} \cdot g_k$$

evidently a complex inner product, whereby $F(i \cdot slS^1)$ becomes a pre-Hilbert space. Hence we may turn $i \cdot slS^1$ into a real pre-Hilbert space, with inner product given by the real part of

$$\langle f, g \rangle_{1/2} = \sum_{k \in \mathbf{N}} k \overline{f_k} \cdot g_k$$

for $f, g \in i \cdot slS^1$. We denote the corresponding real Hilbert space by $\mathcal{H}_r^{1/2}$. Furthermore, we let $\mathcal{H}^{1/2}$ denote the corresponding complex Hilbert space defined by introducing the ordinary complex structure on the Fourier components. We remark that the index and superscript 1/2 used above, serves to indicate the analogy with the Sobolev space of a "half time" differentiable functions (in generalized sense). Observe that we then get the identity $\omega_P(if, ig) = -2i \cdot Im \langle f, g \rangle_{1/2}$.

The spin representation of the special loop group SLS^1, is given by $U_P(e^{if}) = e^{i \cdot \phi(f)}$. It leaves each charge sector $\mathcal{H}_q$ invariant and it fulfils the Weyl form of the canonical commutation relations

$$U_P\left(e^{if}\right) U_P\left(e^{ig}\right) = e^{\frac{1}{2}\omega(if,ig)} U_P\left(e^{i(f+g)}\right)$$

for all $f, g \in \mathcal{H}_r^{1/2}$, as we will show below. Later on we will show that these representations on different charge sectors $\mathcal{H}_q$, are unitarily equivalent.

Recall that $\mathcal{D}$ is a dense set of analytical vectors for $\phi(f) = -i \cdot dU_P(if)$, and since it is closed, by definition, it is selfadjoint. Let $F, G \in \mathcal{D}$ be arbitrarily chosen, then

$$\left\langle \phi(g)G, e^{-i\phi(f)}F \right\rangle = \sum_{n=0}^{\infty} \left\langle \phi(g)G, \frac{(-i)^n}{n!} \phi(f)^n F \right\rangle$$

$$= \sum_{n=0}^{\infty} \left\langle G, \frac{(-i)^n}{n!} (\phi(f)^n \phi(g) + n \cdot \phi(f)^{n-1} \omega_P(if, ig)) F \right\rangle$$

where we have used the selfadjointness of $\phi(g)$ together with the fact that $[\phi(g), \phi(f)^n] = n \cdot \omega_P(if, ig) \cdot \phi(f)^{n-1}$ due to the commutation relations derived above. Consequently

$$
\begin{aligned}
&\left\langle \phi(g)G, e^{-i\phi(f)}F \right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle \frac{i^n}{n!}\phi(f)^n G, \phi(g)F \right\rangle + \left\langle G, -ie^{-i\phi(f)}\omega_P(if, ig)F \right\rangle \\
&= \left\langle G, e^{-i\phi(f)}(\phi(g) - i \cdot \omega_P(if, ig))F \right\rangle
\end{aligned}
$$

Since $e^{-i\phi(f)}$ is bounded, in fact it is unitary, it follows that $\phi(g)^* = \phi(g)$ is well-defined on $e^{-i\phi(f)}F$ and $\phi(g)e^{-i\phi(f)}F = e^{-i\phi(f)}(\phi(g) - i \cdot \omega_P(if, ig) \cdot I)F$ for any $F \in \mathcal{D}$. Thus

$$
\left\| \phi(g)e^{-i\phi(f)}F \right\| \leq \|\phi(g)F\| + |\omega_P(if, ig)| \cdot \|F\|
$$

where the unitarity of $e^{-i\phi(f)}$ has been used. Since $\mathcal{D}$ is a core for $\phi(g)$, i.e. $\overline{\phi(g) \restriction_{\mathcal{D}}} = \phi(g)$, any $F \in \mathcal{D}(\phi(g))$ can be approximated by a sequence $\{F_k\}_{k \in \mathbf{N}} \subset \mathcal{D}$, such that $F_k \to F$ and $\phi(g)F_k \to \phi(g)F$. Then

$$
\begin{aligned}
&\left\| \phi(g)e^{-i\phi(f)}(F_n - F_m) \right\| \\
&= \|\phi(g)(F_n - F_m)\| + |\omega_P(if, ig)| \cdot \|F_n - F_m\| \\
&\to 0 \quad \text{as } m, n \to \infty
\end{aligned}
$$

showing that $\{\phi(g)e^{-i\phi(f)}F_k\}_{k \in \mathbf{N}}$ form a Cauchy sequence and hence converge. Therefore $e^{-i\phi(f)}F \in \mathcal{D}(\phi(g))$ and $\phi(g)e^{-i\phi(f)}F_k \to \phi(g)e^{-i\phi(f)}F$. Thus $\mathcal{D}(\phi(g))$ is invariant under each $e^{-i\phi(f)}$ and the formula $\phi(g)e^{-i\phi(f)} = e^{-i\phi(f)}(\phi(g) - i \cdot \omega_P(if, ig) \cdot I)$ extend to all of $\mathcal{D}(\phi(g))$. Then, for any $F \in \mathcal{D}(\phi(g))$

$$
\begin{aligned}
&\frac{d}{ds} e^{is\phi(f)} e^{is\phi(g)} e^{-is\phi(f+g)} F \\
&= e^{is\phi(f)} e^{is\phi(g)} i \cdot (\phi(f) + \phi(g) - \phi(f+g)) e^{-is\phi(f+g)} F \\
&\quad + e^{is\phi(f)} i \cdot (i \cdot \omega_P(isg, if)) e^{is\phi(g)} e^{-is\phi(f+g)} F \\
&= s \cdot \omega_P(if, ig) e^{is\phi(f)} e^{is\phi(g)} e^{-is\phi(f+g)} F
\end{aligned}
$$

since $\omega_P(ig, if) = -\omega_P(if, ig)$. Notice that all of the products above are well-defined. Integration then gives the following identity between bounded operators

$$e^{i\phi(f)}e^{i\phi(g)}e^{-i\phi(f+g)} = I + \int_0^1 ds\, \omega_P(if, ig) \cdot s \cdot e^{is\phi(f)}e^{is\phi(g)}e^{-is\phi(f+g)}$$

This equation can be solved by iteration as follows. Put $V_0(f,g) = I$ and recursively $V_{n+1}(f,g) = \int_0^1 ds\, \omega_P(if,ig) \cdot s \cdot V_n(sf, sg)$, for $n \in \mathbf{N} \cup \{0\}$. Then

$$V_1(f,g) = \int_0^1 ds\, \omega_P(if,ig) \cdot s \cdot I = \frac{\omega_P(if,ig)}{2} \cdot I$$

Assume that $V_n(f,g) = \left(\frac{\omega_P(if,ig)}{2}\right)^n \cdot \frac{1}{n!} \cdot I$, then $V_n(sf,sg) = s^{2n} \cdot V_n(f,g)$ and

$$\begin{aligned} V_{n+1}(f,g) &= \int_0^1 ds\, \omega_P(if,ig) \cdot s \cdot s^{2n} \cdot V_n(f,g) \\ &= \left(\frac{\omega_P(if,ig)}{2}\right)^{n+1} \cdot \frac{1}{(n+1)!} \cdot I \end{aligned}$$

So induction gives that the formula above holds for all $n \in \mathbf{N} \cup \{0\}$. Define

$$V(f,g) = \lim_{N\to\infty} \sum_{n=0}^{N} V_n(f,g) = e^{\frac{1}{2}\omega_P(if,ig)} \cdot I$$

Hence $V(f,g)$ is the unique solution to the integral equation

$$\begin{aligned} V(f,g) &= \sum_{n=0}^{\infty} V_n(f,g) \\ &= I + \int_0^1 ds \cdot \omega_P(if,ig) \cdot s \cdot \sum_{n=1}^{\infty} V_{n-1}(sf,sg) \\ &= I + \int_0^1 ds \cdot \omega_P(if,ig) \cdot s \cdot V(sf,sg) \end{aligned}$$

whence

$$e^{i\phi(f)}e^{i\phi(g)}e^{-i\phi(f+g)} = V(f,g) = e^{\frac{1}{2}\omega_P(if,ig)} \cdot I$$

and

$$\begin{aligned} e^{i\phi(f)}e^{i\phi(g)} &= e^{\frac{1}{2}\omega_P(if,ig)} \cdot e^{i\phi(f+g)} \\ &= e^{\omega_P(if,ig)} \cdot e^{i\phi(g)}e^{i\phi(f)} \end{aligned}$$

for any $f, g \in i \cdot slS^1$.

We end this treatment of the special loop group by calculating the vacuum functional. The formula $\phi(g)\phi(f)^m = \phi(f)^m\phi(g) + m \cdot \omega_P(if, ig)\phi(f)^{m-1}$, on $\mathcal{D}$, almost holds with $g = f_-$, even though f_- is complex, since it holds for its real and imaginary parts f_-^r and f_-^i, both in $C^1(S^1)_r$, where $f_- = f_-^r + i \cdot f_-^i$. Thus direct calculations give

$$\begin{aligned}
\phi(f_-)\phi(f)^m F &= (\phi(f_-^r) + i \cdot \phi(f_-^i))\phi(f)^m F \\
&= \phi(f)^m(\phi(f_-^r) + i \cdot \phi(f_-^i))F \\
&\quad + m \cdot (\omega_P(if, if_-^r) + i \cdot \omega_P(if, if_-^i))\phi(f)^{m-1}F \\
&= \phi(f)^m\phi(f_-)F + m \cdot \|f\|_{1/2}^2\,\phi(f)^{m-1}F
\end{aligned}$$

where $\|f\|_{1/2}^2 = \langle f, f\rangle_{1/2}$. Especially for $F = \Omega$ we get

$$\phi(f_-)\phi(f)^m\Omega = m \cdot \|f\|_{1/2}^2 \cdot \phi(f)^{m-1}\Omega$$

since $\phi(f_-)\Omega = 0$. Because Ω is an analytic vector for $\phi(f)$, we have

$$\begin{aligned}
\left\langle \Omega, U_P\left(e^{if}\right)\Omega\right\rangle &= \left\langle \Omega, e^{i\phi(f)}\Omega\right\rangle \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}\langle \Omega, \phi(f)^n\Omega\rangle
\end{aligned}$$

We obtain immediately

$$\begin{aligned}
\langle \Omega, \phi(f)^n\Omega\rangle &= \left\langle \Omega, (\phi(f_+) + \phi(f_-))\phi(f)^{n-1}\Omega\right\rangle \\
&= \left\langle \phi(f_-)\Omega, \phi(f)^{n-1}\Omega\right\rangle + \left\langle \Omega, \phi(f_-)\phi(f)^{n-1}\Omega\right\rangle \\
&= 0 + \left\langle \Omega, (n-1) \cdot \|f\|_{1/2}^2 \cdot \phi(f)^{n-2}\Omega\right\rangle \\
&= (n-1) \cdot \|f\|_{1/2}^2 \cdot \left\langle \Omega, \phi(f)^{n-2}\Omega\right\rangle
\end{aligned}$$

Hence by induction we have

$$\langle \Omega, \phi(f)^n\Omega\rangle = 0$$

for $n \in \mathbf{N}$ odd, since $\langle \Omega, \phi(f)\Omega\rangle = 0$, and

$$\begin{aligned}
\langle \Omega, \phi(f)^n\Omega\rangle &= (n-1)(n-3) \cdot \ldots \cdot 1 \cdot \|f\|_{1/2}^n \\
&= \frac{n!}{\left(\frac{n}{2}\right)! \cdot 2^{n/2}} \cdot \|f\|_{1/2}^n
\end{aligned}$$

for $n \in \mathbf{N}$ even, since $\langle \Omega, \Omega \rangle = 1$. Thus

$$\begin{aligned}
\left\langle \Omega, U_P\left(e^{if}\right)\Omega\right\rangle &= \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!}\frac{(2k)!}{k!\cdot 2^k}\cdot \|f\|_{1/2}^{2k} \\
&= \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!}\cdot \left(\|f\|_{1/2}^2\right)^k \\
&= e^{-\frac{1}{2}\cdot\|f\|_{1/2}^2}
\end{aligned}$$

giving the vacuum functional.

We now turn to the charge subgroup C, of the loop group LS^1, generated by elements of the form e^{iF_0}, where $F_0(\theta) = n_F \cdot \theta + f_0$, with $n_F \in \mathbf{Z}$ and $f_0 \in \mathbf{R}$.

The charge group C has infintely many disconnected components, as a subgroup of LS^1, and these components are labelled by an integer n_F, the winding number for e^{iF}.

The shift operator s on $\mathcal{H}$ given by $s(e_k) = e_{k+1}$, i.e. $(sh)(\theta) = e^{i\theta}\cdot h(\theta)$, for any $h \in \mathcal{H}$, is evidently unitary. So $s \in \mathcal{U}(\mathcal{H})$, in fact, $s \in \mathcal{U}_2(\mathcal{H}, P)$ because $[P, s]$ is a Hilbert-Schmidt operator, since $[P, s]e_k = Pe_{k+1} - sPe_k = -\delta_{k+1}\cdot e_0$, so $[P, s]$ is the rank 1 operator $f \to -\langle e_{-1}, f\rangle e_0$. However, the shift operator s is not generated by any element in the Lie algebra $u_2(\mathcal{H}, P)$, since its generator $i\theta$, commuted with P, does not fulfil the Hilbert-Schmidt condition. We will show that $[P, \theta]$ has an infinite Hilbert-Schmidt norm. Let $b(\theta)$ denote the function $\theta \mapsto \theta$, on $[0, 2\pi]$. The Fourier series is given by $b(\theta) = \sum_{k\in\mathbf{Z}} b_k e_k(\theta)$, where $b_0 = \pi$ and $b_k = \frac{i}{k}$,, for $k \neq 0$. Hence

$$\begin{aligned}
[P, \theta]e_n &= P\sum_{k\in\mathbf{Z}} b_k e_{k+n} - \sum_{k\in\mathbf{Z}} b_k e_{k+n}\chi(-n-1) \\
&= \sum_{k\in\mathbf{Z}} b_k e_{k+n}\cdot(\chi(-k-n-1) - \chi(-n-1)) \\
&= \begin{cases} \sum_{k<-n} b_k e_{k+n} & \text{, for } n \geq 0 \\ -\sum_{k\geq -n} b_k e_{k+n} & \text{, for } n < 0 \end{cases}
\end{aligned}$$

where χ denote the indicator function for $\mathbf{N}\cup\{0\}$ defined above. So

$$
\begin{aligned}
\| [P,\theta] \|_{HS}^2 &= \sum_{n\in\mathbf{Z}} \| [P,\theta]e_n \|^2 \\
&= \sum_{n\in\mathbf{N}\cup\{0\}} \sum_{k<-n} |b_k|^2 + \sum_{n\in-\mathbf{N}} \sum_{k\geq -n} |b_k|^2 \\
&= \sum_{n\in\mathbf{N}\cup\{0\}} \sum_{k<-n} \frac{1}{k^2} + \sum_{n\in-\mathbf{N}} \sum_{k\geq -n} \frac{1}{k^2} \\
&= 2\cdot \sum_{n\in\mathbf{N}} \sum_{k\geq n} \frac{1}{k^2} \\
&> 2\cdot \sum_{n\in\mathbf{N}} \int_n^\infty \frac{1}{x^2}dx = 2\cdot \sum_{n\in\mathbf{N}} \frac{1}{n}
\end{aligned}
$$

which is a well-known divergent series. Therefore $[P,\theta]$ is not a Hilbert-Schmidt operator, as claimed. This means that we cannot use the method developed in chapter 1 to find a unitary operator $U_P(s)$, such that

$$a_P(sh) = U_P(s)a_P(h)U_P(s)^{-1}$$

for all $h \in \mathcal{H}$. In this case one can nevertheless explicitely construct such a unitary operator $U_P(s)$. Notice that $a_P(e_k) = a((1-P)e_k) + a^*(\Gamma P e_k) = \chi(k)\cdot a(e_k) + \chi(-k-1)\cdot a^*(e_k)$, so $a_P(e_k) = a(e_k)$, for $k \geq 0$, and $a_P(e_k) = a^*(e_k)$, for $k < 0$, and then $a_P(se_k) = a_P(e_{k+1}) = a(e_{k+1})$, for $k \geq -1$, and $a_P(se_k) = a_P(e_{k+1}) = a^*(e_{k+1})$, for $k < -1$. Hence, our demand to $U_P(s)$ is equivalent to the claim that it has to fulfil

$$U_P(s)a(e_k)U_P(s)^{-1} = a(e_{k+1})$$

for $k \neq -1$ and

$$U_P(s)a(e_{-1})U_P(s)^{-1} = a^*(e_0)$$

Define an operator S on $\mathcal{F}_\wedge(\mathcal{H})$ by its action on product basis vectors $S\Omega = e_0$, $Se_{-1} = \Omega$, $S(e_{k_1}\wedge\ldots\wedge e_{k_n}) = e_{k_1+1}\wedge\ldots\wedge e_{k_n+1}\wedge e_0$, provided each $k_j \neq -1$, for $j = 1,\ldots,n$, and $S(e_{k_1}\wedge\ldots\wedge e_{k_n}\wedge e_{-1}) = e_{k_1+1}\wedge\ldots\wedge e_{k_n+1}$, where each $k_j \neq -1$, for $j = 1,\ldots,n$. Extension by linearity and continuity gives a bounded operator S on $\mathcal{F}_\wedge(\mathcal{H})$. From above it follows that S is a surjective isometry, whence it is unitary. Let Q be the charge operator on $\mathcal{F}_\wedge(\mathcal{H})$, with domain $\mathcal{D}$, defined in section 1.5. From the action of S on product basis vectors, which belongs to $\mathcal{D}$, we immediately see that S maps the q'th charge sector $\mathcal{H}_q$ into the $(q+1)$'th charge sector $\mathcal{H}_{q+1}$. Then $(QS - SQ)(e_{k_1} \wedge \ldots \wedge e_{k_n}) = (q+1-q)S(e_{k_1} \wedge \ldots \wedge e_{k_n})$, where $e_{k_1} \wedge \ldots \wedge e_{k_n}$ denote an arbitrary product basis vector in $\mathcal{H}_q$ (it also holds on $\Omega \in \mathcal{H}_0$). Then we get the following commutation relation $[Q, S] = S$ on $\mathcal{D}$, or equivalently

$$QS = S(Q+I)$$

By direct calculation it follows that S satisfies our demand to $U_P(s)$ and we may put $U_P(s) = S$. Hence we have constructed the unitary operator $U_P(s)$ such that $a_P(sh) = U_P(s)a_P(h)U_P(s)^{-1}$, for all $h \in \mathcal{H}$. Notice that s^n acts as the multiplication operator $e^{in\theta}$. We therefore define $U_P(s^n) = S^n$, hence $a_P(s^n h) = S^n a_P(h)(S^{-1})^n = U_P(s^n)a_P(h)U_P(s^n)^{-1}$, for all $h \in \mathcal{H}$ and $n \in \mathbf{Z}$. Then we have handle the first term $n_F \cdot \theta$ in F_0. We thus turn to the second term f_0 in $F_0 = n_F \cdot \theta + f_0$.

Consider e^{if_0}, with $f_0 \in \mathbf{R}$, which trivially belongs to the restricted unitary group $\mathcal{U}_2(\mathcal{H}, P)$. Then the unitary operator $U_P(e^{if_0})$ on $\mathcal{F}_\wedge(\mathcal{H})$ is explicitely given by $U_P(e^{if_0}) = e^{dU_P(if_0)} = e^{f_0 \cdot dU_P(iI)} = e^{if_0 Q}$, where Q is the change operator, discussed in section 1.5.

Combining the above discussion of the two terms in F_0, we are lead to define $U_P(e^{iF_0})$ by

$$U_P\left(e^{iF_0}\right) = e^{\frac{1}{2}if_0 Q} S^{n_F} e^{\frac{1}{2}if_0 Q}$$

Then $U_P\left(e^{iF_0}\right)$ is unitary and $U_P(e^{iF_0})a_P(h)U_P(e^{iF_0})^{-1} = a_P(e^{iF_0}h)$, as it should be. We will now calculate an explicit formula for the associated cocycle. For arbitrary $n \in \mathbf{N} \cup \{0\}$, is $QS^n = S^n(Q + n \cdot I)$ and $Q^k S^n = S^n(Q + n \cdot I)^k$ on $\mathcal{D}$. Then $e^{\lambda Q} S^n = S^n e^{\lambda(Q + n \cdot I)}$ on $\mathcal{D}$, for any $n \in \mathbf{N} \cup \{0\}$, hence any $n \in \mathbf{Z}$, and any $\lambda \in \mathbf{C}$, since $\mathcal{D}$ is a set of analytic vectors for Q (see section 1.5), which is invariant under S. Hence

$$e^{\frac{1}{2}if_0 Q} S^{n_G} = S^{n_G} e^{\frac{1}{2}if_0(Q + n_G)} = S^{n_G} e^{\frac{1}{2}if_0 Q} e^{\frac{1}{2}if_0 n_G}$$

and then

$$S^{n_F} e^{\frac{1}{2}ig_0 Q} = e^{\frac{1}{2}ig_0 Q} S^{n_F} e^{-\frac{1}{2}ig_0 n_F}$$

which implies that

$$\begin{aligned}
U_P\left(e^{iF_0}\right)U_P\left(e^{iG_0}\right) &= e^{\frac{1}{2}if_0Q}S^{n_F}e^{\frac{1}{2}if_0Q}e^{\frac{1}{2}ig_0Q}S^{n_G}e^{\frac{1}{2}ig_0Q} \\
&= e^{\frac{1}{2}if_0Q}e^{\frac{1}{2}ig_0Q}e^{-\frac{1}{2}ig_0n_F}S^{n_F}S^{n_G}e^{\frac{1}{2}if_0Q}e^{\frac{1}{2}ig_0Q}e^{\frac{1}{2}if_0n_G} \\
&= e^{\frac{1}{2}i(f_0n_G-g_0n_F)}e^{\frac{1}{2}i(f_0+g_0)Q}S^{(n_F+n_G)}e^{\frac{1}{2}i(f_0+g_0)Q} \\
&= c\left(e^{iF_0},e^{iG_0}\right)\cdot U_P\left(e^{i(F_0+G_0)}\right)
\end{aligned}$$

whereof we see that the cocycle is given by

$$c\left(e^{iF_0},e^{iG_0}\right)=e^{\frac{1}{2}i(f_0n_G-g_0n_F)}$$

Thus we have constructed a projective unitary representation of the abelian charge group C.

We will now show that $U_P(e^{if})$ and $U_P(e^{iF_0})$ commute for all $if\in slS^1$ and $e^{iF_0}\in C$. This is a consequence of the fact that the unitary operators $V_1=Se^{i\phi(f)}$ and $V_2=e^{i\phi(f)}S$ implement the same automorphism. We have

$$\begin{aligned}
\alpha_1(a_P(g)) &= V_1a_P(g)V_1^*=U_P(s)a_P\left(e^{if}g\right)U_P(s) \\
&= a_P\left(e^{i\theta}e^{if}g\right)=a_P\left(e^{i(f+\theta)}g\right)
\end{aligned}$$

and that

$$\begin{aligned}
\alpha_2(a_P(g)) &= V_2a_P(g)V_2^*=U_P\left(e^{if}\right)a_P(sg)U_P\left(e^{if}\right)^{-1} \\
&= a_P\left(e^{if}e^{i\theta}g\right)=a_P\left(e^{i(f+\theta)}g\right)
\end{aligned}$$

so $\alpha_1(a_P(g))=\alpha_2(a_P(g))$. Since the representation of the CAR-algebra, labelled by P, is irreducible, it follows that $V_1=c_0(V_1,V_2)V_2$, where $c_0(V_1,V_2)\in\mathbf{C}$ and has absolute value 1, due to the unitarity of V_1 and V_2. That is

$$Se^{i\phi(f)}=c_0(f)\cdot e^{i\phi(f)}S$$

with $|c_0(f)|=1$, for all $f\in\mathcal{H}$. We write $c_0(f)$ as $e^{ic(f)}$, with $c(f)$ real. Then the unitarity of S gives

$$e^{i(\phi(f)+c(f))}=Se^{i\phi(f)}S^{-1}=e^{iS\phi(f)S^{-1}}$$

on $\mathcal{D}$, since S leaves $\mathcal{D}$ invariant and $\mathcal{D}$ consists of analytic vectors for $\phi(f)$. Put $f=t\cdot g$ and take the derivative at $t=0$ to get

$$\phi(g)+c(g)\cdot I=S\phi(g)S^{-1}$$

where we have used that $t \to c(tg)$ is linear. By taking the vacuum expectation value of the equation

$$c(g) \cdot I = \phi(g) - S^{-1}\phi(g)S$$

we get

$$c(g) = \langle \Omega, \phi(g)\Omega \rangle - \langle S\Omega, \phi(g)S\Omega \rangle = 0 - \langle e_0, \phi(g)e_0 \rangle$$

by (1.10). Then

$$\begin{aligned} c(g) &= -\langle e_0, idU_P(ig)e_0 \rangle \\ &= i \cdot \langle e_0, ((I-P)ig(I-P) + \Gamma PigP\Gamma)e_0 \rangle \end{aligned}$$

since it is only the linear part of $(ig)_P$ which contribute, due to $dU(((ig)_P)_2)e_0 \in \wedge^3\mathcal{H} \perp \mathcal{H}$, and $dU(((ig)_P)_1)$ is given by the Foch-Cook generator. Hence

$$\begin{aligned} c(g) &= -\langle (I-P)e_0, g(I-P)e_0 \rangle + \langle ig\Gamma Pe_0, \Gamma Pe_0 \rangle \\ &= -\langle e_0, ge_0 \rangle + 0 \\ &= 0 \end{aligned}$$

for any $g \in \mathcal{H}$, since $g_0 = \langle e_0, ge_0 \rangle = 0$. Whereby

$$\phi(f)S = S\phi(f)$$

on $\mathcal{D}$, and

$$e^{i\phi(f)}S = Se^{i\phi(f)}$$

for any $f \in \mathcal{H}$. Thus

$$\begin{aligned} U_P\left(e^{if}\right) U_P\left(e^{iF_0}\right) &= e^{i\phi(f)} e^{\frac{1}{2}if_0Q} S^{n_F} e^{\frac{1}{2}if_0Q} \\ &= e^{\frac{1}{2}if_0Q} S^{n_F} e^{\frac{1}{2}if_0Q} e^{i\phi(f)} \\ &= U_P\left(e^{iF_0}\right) U_P\left(e^{if}\right) \end{aligned}$$

due to the fact that $[Q, e^{i\phi(f)}] = 0$, on $\mathcal{D}$, shown in section 1.5. Hence we put $U_P(e^{iF}) = U_P(e^{iF_0})U_P(e^{if})$, which is the declared projective representation of the loop group LS^1. Notice that all representations of SLS^1, in the different charge sectors, are unitarily equivalent, $S^{-1}\phi(f)|_{\mathcal{H}_q} S = \phi(f)|_{\mathcal{H}_{q+1}}$, since $S : \mathcal{H}_q \to \mathcal{H}_{q+1}$ is unitary.

We have constructed a represenation of the canonical commution relation in each charge sector $\mathcal{H}_q$, by use of the mapping $f \to \phi(f)$, on the anti-symmetric Fock Hilbert space. All these representations are shown to be unitarily equivalent and are moreover unitarily equivalent to the Fock representations $f \to \pi(f)$ in the symmetric Fock Hilbert space (modelled over $\mathcal{H}^{1/2}$), discussed in section 2.1. However, we delay the proof of the last claimed equivalence until section 4.4.

4.2 The diffeomorphism group $\mathit{Diff}^+(S^1)$ as a unitary group.

In this section we will study the group consisting of orientation preserving diffeomorphisms of the unit circle S^1. We will do this by realizing $\mathit{Diff}^+(S^1)$ as a subgroup of the restricted unitary group, using the spin representation on a Lie algebra level. It turns out that we get representations of the Virasoro algebra in terms of the spin representation, whereby the connection of chapter 1 and chapter 3 appear. However, this is not the only possible realization of $\mathit{Diff}^+(S^1)$.

In section 4.3 we will realize $\mathit{Diff}^+(S^1)$ as a subgroup of the restricted symplectic group, using the metaplectic representation on a Lie algebra level. Thereby we construct positive energy projective representations of the Virasoro algebra. The Lie algebra consists of unbounded operators, which complicates the subject. We consider therefore only relative simple problems. From this also the connection between the metaplectic representation and the Virasoro algebra will becomme clear.

There is a large number of litterature on the Virasoro algebra, however, the following is mostly based on section 1.5, 2.3 and 3.2 of this thesis together with [P-S], [Ne] and [Lu 2], but also [K-R] and [Mi] is of interest for this subject.

Let $\mathit{Diff}^+(S^1)$ denote the orientation preserving diffeomorphism of the unit circle S^1, i.e. an element $\psi \in \mathit{Diff}^+(S^1)$ is of the form $\psi\left(e^{i\theta}\right) = e^{i\cdot\phi(\theta)}$, where $e^{i\theta} \in S^1$, ϕ is a smooth real function such that $\phi(\theta + 2\pi) = \phi(\theta) + 2\pi$ and $\phi'(\theta) > 0$.

In this section we consider the case where $\mathit{Diff}^+(S^1)$ is realized as a subgroup of the restricted unitary group.

The diffeomorphism group $\mathit{Diff}^+(S^1)$ can act on $\mathcal{H} = L_2(S^1)$ in more than one way ([P-S, p.91] and [Ne, p. 411]). The action becomes unitary if we choose it as follows

$$(u_\phi f)(\theta) = f(\phi(\theta)) \cdot |\phi'(\theta)|^{\frac{1}{2}}$$

for any $f \in \mathcal{H}$. Notice that $|\phi'(\theta)| = \phi'(\theta)$, so we may omitte the absolut value symbol if we want to. Evidently

$$\begin{aligned}\langle u_\phi f, u_\phi g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \overline{f(\phi(\theta))}g(\phi(\theta))\cdot|\phi'(\theta)|\,d\theta \\ &= \frac{1}{2\pi}\int_{\phi(0)}^{\phi(0)+2\pi} \overline{f(\tilde\theta)}g(\tilde\theta)d\tilde\theta \\ &= \langle f, g\rangle\end{aligned}$$

for all $f, g \in \mathcal{H}$, since $\phi(\theta+2\pi) = \phi(\theta)+2\pi$ and $\phi'(\theta) > 0$. So u_ϕ is indeed unitary, i.e. $u_\phi \in \mathcal{U}(\mathcal{H})$. Notice that the corresponding diffeomorphism in $\mathit{Diff}^+(S^1)$ is given by $\psi(e^{i\theta}) = e^{i\phi^{-1}(\theta)}$. It follows that $\psi \to u_\phi$ defines an anti-representation of $\mathit{Diff}^+(S^1)$ since

$$\begin{aligned}(u_{\phi_2}(u_{\phi_1}(f)))(\theta) &= u_{\phi_1}(f(\phi_2(\theta)))\cdot|\phi_2'(\theta)|^{\frac{1}{2}} \\ &= f(\phi_1(\phi_2(\theta)))\cdot|\phi_1'(\phi_2'(\theta))\phi_2'(\theta)|^{\frac{1}{2}} \\ &= f((\phi_1\circ\phi_2)(\theta))\cdot|(\phi_1\circ\phi_2)'(\theta)|^{\frac{1}{2}} \\ &= (u_{\phi_1\circ\phi_2}(f))(\theta)\end{aligned}$$

Introduce the splitting, or polarization, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, as in section 4.1, when considering the loop group LS^1, and let P be the orthogonal projection onto $\mathcal{H}_-$. It follows that $u_\phi \in \mathcal{U}_2(\mathcal{H}, P)$. That is, we have to prove that $[P, u_\phi]$ is a Hilbert-Schmidt operator.

The action of u_ϕ is represented by the kernel $\delta(\phi(\theta)-\alpha)\cdot\phi'(\theta)^{\frac{1}{2}}$, where δ denote the Dirac delta function

$$\int_0^{2\pi}\delta(\phi(\theta)-\alpha)\cdot\phi'(\theta)^{\frac{1}{2}}f(\alpha)d\alpha = f(\phi(\theta))\cdot\phi'(\theta)^{\frac{1}{2}}$$

The polarization operator J on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, given by $Jf_\pm = \pm f_\pm$, for $f_\pm \in \mathcal{H}_\pm$, is represented by the singular integral operator

$$(Jf)(\theta) = \frac{1}{2\pi}PV\int_0^{2\pi} K(\theta,\varphi)f(\varphi)d\varphi$$

where PV denote the principal value of the integral

$$PV\int_0^{2\pi} K(\theta,\varphi)f(\varphi)d\varphi = \lim_{\varepsilon\to 0}\left(\int_0^{\theta-\varepsilon} + \int_{\theta+\varepsilon}^{2\pi}\right)K(\theta,\varphi)f(\varphi)d\varphi$$

and the Kernel $K(\theta,\varphi)$ is given by

$$
\begin{aligned}
K(\theta,\varphi) &= \sum_{k\geq 0} e^{ik(\theta-\varphi)} - \sum_{k<0} e^{ik(\theta-\varphi)} \\
&= 1 + i\cdot\cot\frac{1}{2}(\theta-\varphi)
\end{aligned}
$$

which clearly is singular on the diagonal $\theta=\varphi$. Then the kernel $K_{[u_\varphi,J]}$ of the commutator $[u_\varphi,J]$ is given by

$$
\int_0^{2\pi} d\alpha\,\left(\delta(\phi(\theta))-\alpha)\cdot\phi'(\theta)^{\frac{1}{2}}\cdot K(\alpha,\beta) - K(\theta,\alpha)\cdot\delta(\phi(\alpha))-\beta)\cdot\phi'(\alpha)^{\frac{1}{2}}\right)
$$

$$
= \phi'(\theta)^{\frac{1}{2}}\cdot K(\phi(\theta),\beta) - K(\theta,\phi^{-1}(\beta))\cdot(\phi^{-1})'(\beta)^{\frac{1}{2}}
$$

since

$$
\begin{aligned}
&([u_\phi,J]f)(\theta) \\
&= \int_0^{2\pi} d\alpha\,\left(\delta(\phi(\theta)-\alpha)\cdot\phi'(\theta)^{\frac{1}{2}}\left(\frac{1}{2\pi}PV\int_0^{2\pi} K(\alpha,\beta)f(\beta)d\beta\right)\right) \\
&\quad -\frac{1}{2\pi}PV\int_0^{2\pi} d\alpha\,\left(K(\theta,\alpha)\left(\int_0^{2\pi}\delta(\phi(\alpha)-\beta)\phi'(\alpha)f(\beta)d\beta\right)\right) \\
&= \frac{1}{2\pi}PV\int_0^{2\pi} d\beta\int_0^{2\pi} d\alpha \\
&\quad \left(\delta(\phi(\theta)-\alpha)\phi'(\theta)^{\frac{1}{2}}K(\alpha,\beta) - K(\theta,\alpha)\delta(\phi(\alpha)-\beta)\phi'(\alpha)^{\frac{1}{2}}\right)f(\beta)
\end{aligned}
$$

Notice that $K(\theta,\varphi)=1+i\cdot\cot\frac{1}{2}(\theta-\varphi)$ is indeed a smooth function in both variables except possibly on the diagonal, in fact

$$
K(\theta,\varphi) = \frac{2\cdot i}{\theta-\varphi} + S(\theta,\varphi)
$$

where $S(\theta,\varphi)=S_1(\frac{1}{2}(\theta-\varphi))$ is the smooth function

$$
S_1(x) = 1 - \frac{i}{3}x - \frac{i}{45}x^3 - \dots
$$

Evidently is $K_{[u_\varphi,J]}$ smooth off the diagonal, $\theta\neq\varphi$, however, $K_{[u_\varphi,J]}$ is smooth on all of $S^1\times S^1$. We only need the continuity of $K_{[u_\varphi,J]}$ across the diagonal, to guarantee that the kernel $K_{[u_\varphi,J]}$ for $[u_\varphi, J]$ becomes square integrable on $S^1 \times S^1$. By a Taylor expansion to first order in $(\phi^{-1})'(\beta)$, for β sufficiently close to $\phi(\theta)$, it easily follows that

$$\begin{aligned}
&\phi'(\theta)^{\frac{1}{2}} \cdot K(\phi(\theta),\beta) - K(\theta,\phi^{-1}(\beta)) \cdot (\phi^{-1})'(\beta)^{\frac{1}{2}} \\
&= 2i \cdot \left(\frac{\phi'(\theta)^{\frac{1}{2}}}{\phi(\theta)-\beta} - \frac{(\phi^{-1})'(\beta)^{\frac{1}{2}}}{\theta - \phi^{-1}(\beta)} \right) \\
&\quad + \phi'(\theta)^{\frac{1}{2}} \cdot S(\phi(\theta),\beta) - S(\theta,\phi^{-1}(\beta)) \cdot (\phi^{-1})'(\beta)^{\frac{1}{2}} \\
&= 2i \cdot \frac{(\phi^{-1})''(\theta)}{(\phi^{-1})'(\theta)} \\
&\quad + O(\beta - \phi(\theta)) + \phi'(\theta)^{\frac{1}{2}} \cdot S(\phi(\theta),\beta) - S(\theta,\phi^{-1}(\beta)) \cdot (\phi^{-1})'(\beta)^{\frac{1}{2}}
\end{aligned}$$

which clearly is a bounded continous function for $|\beta - \phi(\theta)|$ sufficiently small. Here $O(x^n)$ denotes terms in x of order $n \in \mathbf{N}$. This means that the operator represented by the kernel for $[u_\phi, J]$, which naturally is $[u_\phi, J]$, is a Hilbert-Schmidt operator, since its kernel is square integrable on $S^1 \times S^1$. Furthermore, since $[P, u_\phi] = -[u_\phi, \frac{1}{2}(I-J)] = \frac{1}{2}[u_\phi, J]$ it follows that $[P, u_\phi]$ is a Hilbert-Schmidt operator, as claimed.

The associated Lie algebra $\mathit{diff}^+(S^1)$ on $L_2(S^1)$, which is given as the real span of the basis vectors

$$d_k^- = \cos(k\cdot\theta)\cdot d_0 - \frac{1}{2}\cdot k \cdot \sin(k\cdot\theta)$$

$k \in \mathbf{Z}$ and

$$d_k^+ = \sin(k\cdot\theta)\cdot d_0 + \frac{1}{2}\cdot k \cdot \cos(k\cdot\theta)$$

$k \in \mathbf{Z}\setminus\{0\}$, where

$$d_0 = \frac{d}{d\theta}$$

in generalized sense in $L_2(S^1)$. This is in fact a realization of the Lie algebra $\mathit{Vect}(S^1)$ discussed in section 3.2, but with a different choice of basis.

However, it is a bit easier to work with the complexified Lie algebra $\mathit{diff}^+(S^1)_{\mathbf{C}}$ given by the choice of basis vectors

$$d_k = e_k \cdot d_0 + \frac{1}{2} ik \cdot e_k = e_k \left(d_0 + \frac{1}{2} ik \right)$$

where $k \in \mathbf{Z}$ and d_0 is as above. Notice that these (basis vector as) operators in $L_2(S^1)$ are all unbounded, but all with the same maximal domain given by $\mathcal{D}_{\max} = \left\{f \in L_2(S^1) : \sum_{n\in\mathbf{Z}} n^2 \cdot |f_n|^2 < \infty\right\} = \{f \in L_2(S^1) : f' \in L_2(S^1)\}$, where f_n, as usual, denote the n'th Fourier component of f. Moreover, $d_k^* = -d_{-k}$, since it holds on basis vectors e_m and e_n in $L_2(S^1)$, for arbitrary $m, n \in \mathbf{Z}$

$$\begin{aligned}\langle e_m, d_k e_n\rangle &= i\left(n + \frac{1}{2}k\right) \cdot \delta_{m-k-n} = i\left(m - \frac{1}{2}k\right) \cdot \delta_{m-k-n} \\ &= \langle -d_{-k} e_m, e_n\rangle\end{aligned}$$

Observe that $d_k^- = \frac{1}{2}(d_k - d_k^*)$ and $d_k^+ = \frac{1}{2i}(d_k + d_k^*)$. Furthermore we get the following commutation relation

$$[d_m, d_n] = -i(m-n) \cdot d_{m+n}$$

on $\mathcal{D}_{\max}$, which can be verified by a straight forward calculation. This relation is rather close to the corresponding one for the complexification $\mathfrak{d}$ of $Vect(S^1)$, see section 3.2. Finally we have that the d_k's, $k \in \mathbf{Z}$, fulfil the Jacobi identity (again by a straight forward calculation). So the algebra spanned by d_k, $k \in \mathbf{Z}$, is indeed a Lie algebra.

Let $u_2(\mathcal{H}, P)_{\mathbf{C}}$ denote the complexification of $u_2(\mathcal{H}, P)$. We will show that d_k belongs to the enlarged Lie algebra of $u_2(\mathcal{H}, P)_{\mathbf{C}}$, allowing unbounded operators, i.e. that $[P, d_k]$ is Hilbert-Schmidt or equivalently $Pd_{-k}(I-P)d_kP$ is of trace-class, for all $k \in \mathbf{Z}$. However, since we later on will need the more general fact, that $Pd_{-k}(I-P)d_mP$ is of trace-class, for all $k, m \in \mathbf{Z}$, and then especially also for $m = k$, we will show this. Let χ denote the indicator function for $\mathbf{N} \cup \{0\}$, as earlier. Then $d_m P e_n = \chi(-n-1) \cdot i(n + \frac{m}{2}) e_{n+m}$ and

$$\begin{aligned}&d_{-k}(I-P)d_m P e_n = \\ &\qquad \chi(-n-1)\chi(n+m) \cdot i^2 \left(n + \frac{m}{2}\right)\left(n + m - \frac{k}{2}\right) e_{n+m-k}\end{aligned}$$

giving that

$$
\begin{aligned}
&\langle e_n, Pd_{-k}(I-P)d_mPe_n\rangle \\
&= -\left(n+\frac{m}{2}\right)\left(n+m-\frac{k}{2}\right)\cdot\chi(-n-1)\cdot\chi(n+m)\cdot\langle e_n, e_{n+m-k}\rangle \\
&= -\left(n+\frac{k}{2}\right)^2\cdot\chi(-n-1)\cdot\chi(n+k)\cdot\delta_{m-k}
\end{aligned}
$$

for arbitrary $k, m, n \in \mathbf{Z}$. Hence

$$
\begin{aligned}
\mathrm{Tr}(Pd_{-k}(I-P)d_mP) &= \sum_{n\in\mathbf{Z}}\langle e_n, Pd_{-k}(I-P)d_mPe_n\rangle \\
&= -\sum_{n=-k}^{-1}(n+\frac{k}{2})^2\cdot\delta_{m-k}\cdot\chi(k) \\
&= -\delta_{m-k}\cdot\chi(k)\cdot\sum_{n=1}^{k}\left(n-\frac{k}{2}\right)^2 \\
&= -\frac{1}{12}\cdot(k^3+2k)\cdot\delta_{m-k}\cdot\chi(k)
\end{aligned}
$$

which obviously is finite, for all $k, m \in \mathbf{Z}$, and only non-zero, and then negative, for $m = k > 0$, proving the claim. Hence $[P, d_k]$ is Hilbert-Schmidt and considerations analogous to those in section 1.5 then gives that $[P, e^{d_k^\pm}]$ are Hilbert-Schmidt.

The spin representation of the (up till now only real) Lie algebra, provide us with a positive energy representation of the Virasoro algebra (see section 3.2). It is of positive energy due to the fact that $H = -idU_P(d_0)$ is non-negative, since $H\Omega = 0$ and on an arbitrary product basis vector $e_{k_1}\wedge\ldots\wedge e_{k_n}$ is

$$
H\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\right) = \sum_{j=1}^{n}|k_j|\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\right)
$$

because $-i(d_0)_P e_{k_j} = -id_0(1-P)e_{k_j} + id_0Pe_{k_j} = |k_j|\cdot e_{k_j}$. Since H evidently is an unbounded operator in $\mathcal{F}_\wedge(\mathcal{H})$ we have to specify the domain. Obviously its maximal domain are those vectors $F \in \mathcal{F}_\wedge(\mathcal{H})$, where $\|HF\| < \infty$, we call these vectors for finite energy vectors. We choose the algebraic direct sum of all the one-dimensional spaces spanned by the product basis vectors and Ω as the domain $\mathcal{D}(H)$ of H, evidently $\mathcal{D}(H) \subset \mathcal{D}$. As in section 3.3 this give us a natural $\mathbf{N} \cup \{0\}$-grading of $\mathcal{F}_\wedge(\mathcal{H})$

$$\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{m=0}^{\infty} B_m$$

where B_m is the energy eigenspace corresponding to the eigenvalue $m = \sum_{j=1}^{n} |k_j|$. Of course the domain $\mathcal{D}(H)$ is dense in $\mathcal{F}_\wedge(\mathcal{H})$. Moreover $\mathcal{D}(H)$ is the algebraic direct sum of the B_m. Notice that H is zero only on span$\{\Omega, e_0\}$. This construction is very similar to that of the special loop group.

We now complexify the mapping $B \mapsto dU_P(B)$ which until now only is defined for skew-selfadjoint (real linear) elements $B \in u_2(\mathcal{H}, P)$. Since an arbitrary $A \in u_2(\mathcal{H}, P)_{\mathbf{C}}$ may be decomposed as $A = A^- + iA^+$, with $A^- = \frac{1}{2}(A - A^*)$ and $A^+ = \frac{1}{2i}(A + A^*)$ both in $u_2(\mathcal{H}, P)$, we define the complexification of the mapping $B \mapsto dU_P(B)$ by putting $dU_P(A)_{\mathbf{C}} = dU_P(A^-) + i \cdot dU_P(A^+)$, where $A = A^- + i \cdot A^+$. Then we may define the unbounded operator $D_k = dU_P(d_k)_{\mathbf{C}}$. Of course one have to specify the domains explicitely.

A direct calculation shows that

$$D_k\left(e_{k_1} \wedge \ldots \wedge e_{k_n}\right) = \\ \sum_{j=1}^{n} i\left(k_j + \frac{1}{2}k\right) \operatorname{sign}(k_j + k)\left(e_{k_1} \wedge \ldots \wedge e_{k_{j-1}} \wedge e_{k_{j+k}} \wedge e_{k_{j+1}} \wedge \ldots \wedge e_{k_n}\right)$$

So

$$\begin{aligned} \left\|D_k\left(e_{k_1} \wedge \ldots \wedge e_{k_n}\right)\right\| &\leq \left(\sum_{j=1}^{n} |k_j| + \frac{1}{2}n\,|k|\right) \cdot \left\|e_{k_1} \wedge \ldots \wedge e_{k_n}\right\| \\ &= \left\|\left(H + \frac{1}{2}n\,|k|\right)\left(e_{k_1} \wedge \ldots \wedge e_{k_n}\right)\right\| \end{aligned}$$

where the upper bound can be reached, for example when $n = 1$ and $k_1 = k \in \mathbf{N}$. So the maximal domains for all D_k on $\mathcal{F}_\wedge(\mathcal{H})$ is equal to each other and equals the set of finite energy vectors, since $H = -iD_0$. As common domain for all the D_k, $k \in \mathbf{Z}$, we choose $\mathcal{D}(H)$.

Thus $D_k^* = -D_{-k}$ on $\mathcal{D}(H)$ by use of the above complexified mapping and the facts that $d_{-k}^- = d_k^-$ and $d_{-k}^+ = -d_k^+$. Hence, by use of the trace formula derived above, we get

$$\begin{aligned}
\langle D_k\Omega, D_m\Omega\rangle &= \langle\Omega, -D_{-k}D_m\Omega\rangle \\
&= -\langle\Omega, dU_P(d_{-k})_{\mathbf{C}} dU_P(d_m)_{\mathbf{C}}\Omega\rangle \\
&= -\mathrm{Tr}\,(Pd_{-k}(1-P)d_m P) \\
&= \frac{1}{12}\left(k^3+2k\right)\delta_{m-k}\cdot\chi(k)
\end{aligned}$$

where we have used the complexification of formula (1.15) in section 1.5, which can be done by first complexifying in the A-argument,

$$\begin{aligned}
&\langle\Omega, dU_P(A)_{\mathbf{C}} dU_P(B)\Omega\rangle \\
&= \left\langle\Omega, dU_P(A^-)dU_P(B)\Omega\right\rangle + i\cdot\left\langle\Omega, dU_P(A^+)dU_P(B)\Omega\right\rangle \\
&= \mathrm{Tr}\left(PA^-(I-P)BP\right) + i\cdot\mathrm{Tr}\left(PA^+(I-P)BP\right) \\
&= \mathrm{Tr}\left(P(A^-+iA^+)(I-P)BP\right) \\
&= \mathrm{Tr}\,(PA(I-P)BP)
\end{aligned}$$

and the quite similarly complexify in the B-argument, whence the formula follows. Recall that $\chi(k)=1$ for $k\in\mathbf{N}\cup\{0\}$, and that $\chi(k)=0$ for $k\in-\mathbf{N}$, thus $D_k\Omega=0$ for all $k\le 0$, since

$$\|D_k\Omega\|^2 = \frac{1}{12}\left(k^3+2k\right)\cdot\chi(k)$$

Then the associated Lie algebra cocycle becomes

$$\begin{aligned}
\omega(d_k,d_m)_{\mathbf{C}} &= \mathrm{Tr}(Pd_k(I-P)d_mP) - \mathrm{Tr}(Pd_m(I-P)d_kP) \\
&= \frac{1}{12}\left(k^3+2k\right)\left(\delta_{k+m}\cdot\chi(-k)+\delta_{k+m}\cdot\chi(k)\right) \\
&= \frac{1}{12}\left(k^3+2k\right)\delta_{k+m}
\end{aligned}$$

where we have used the complexification of formula (1.16) in section 1.5, again due to the linearity in the arguments and the earlier derived trace formula. Then the commutation relations of the D_k's is given by

$$\begin{aligned}
[D_k, D_m] &= [dU_P(d_k)_{\mathbf{C}}, dU_P(d_m)_{\mathbf{C}}] \\
&= dU_P\left([d_k,d_m]\right)_{\mathbf{C}} + \omega(d_k,d_m)_{\mathbf{C}}\cdot I \\
&= -i(k-m)D_{k+m} + \frac{1}{12}\left(k^3+2k\right)\delta_{k+m}\cdot I
\end{aligned}$$

where we have complexified the formula (1.16) in section 1.5 (or formula (1.13) in section 1.4), due to the linearity in the arguments.

It is this commutation relation which shows us that the representation is rather close to be a representation of the Virasoro algebra studied in section 3.2. In fact we can add a particular constant h to the energy operator H such that the cocycle takes the form $\frac{c}{12}k\,(k^2-1)$, where c is a real constant, and hence we get a representation of the Virasoro algebra. Since the cocycle vanish for $m \neq -k$, we only need to consider the case $m = -k$, whence the commutation relation may be written as

$$[D_k, D_{-k}] = -i2kD_0 + \frac{1}{12}\left(k^3 + 2k\right) = 2kH + \frac{1}{12}\left(k^3 + 2k\right)$$

Let $H_h = H + h$. Then we may rewrite the commutation relation as follows

$$\begin{aligned}[D_k, D_{-k}] &= 2k(H+h) - 2kh + \frac{1}{12}\left(k^3 + 2k\right) \\ &= 2kH_h + \frac{1}{12}k\left(k^2 + 2(1-12h)\right)\end{aligned}$$

For $h = \frac{1}{8}$ we get

$$[D_k, D_{-k}] = 2kH_h + \frac{1}{12}k\left(k^2 - 1\right)$$

and the associated cocycle $\omega_{\frac{1}{8}}(d_k, d_m)$ is

$$\omega_{\frac{1}{8}}(d_k, d_m) = \frac{1}{12}k\left(k^2 - 1\right) \cdot \delta_{k+m}$$

and obviously $c = 1$. Hence our representation is labelled by the pair $(h, c) = \left(\frac{1}{8}, 1\right)$, because $\Omega \in \mathcal{D}(H)$.

Before ending this discussion of $Diff^+(S^1)$ as a unitary group, we will analyse the energy operator H, and thereby H_h, by use of the unitary operator S, defined in section 4.1, which raises the charge number of each charge-eigenvectors by one.

It follows that $[H, S] = SQ$, on $\mathcal{D}(H)$. This is trivial on Ω and e_{-1}. For $-1 \notin \{k_1, \ldots, k_n\}$ is

$$
\begin{aligned}
&[H,S](e_{k_1}\wedge\ldots\wedge e_{k_n})\\
&= H\left(e_{k_1+1}\wedge\ldots\wedge e_{k_n+1}\wedge e_0\right) - S\sum_{j=1}^{n}|k_j|\, e_{k_1}\wedge\ldots\wedge e_{k_n}\\
&= \left(\sum_{j=1}^{n}|k_j+1| + 0 - \sum_{j=1}^{n}|k_j|\right) S\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\right)\\
&= \sum_{j=1}^{n} q_j\cdot S\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\right)\\
&= SQ\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\right)
\end{aligned}
$$

since each k_j has the same sign as k_j+1 (the sign of 0 is $+$ by definition), so $|k_j+1|-|k_j| = \pm 1 = q_j$, for $\mathrm{sign}(k_j) = \pm 1$, where q_j is the charge of e_{k_j}. For $k_{n+1} = -1$, is

$$
\begin{aligned}
&[H,S](e_{k_1}\wedge\ldots\wedge e_{k_n}\wedge e_{-1})\\
&= H\left(e_{k_1+1}\wedge\ldots\wedge e_{k_n+1}\right) - \left(\sum_{j=1}^{n}|k_j|+1\right) S\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\wedge e_{-1}\right)\\
&= \left(\sum_{j=1}^{n}|k_j+1| + 0 - \sum_{j=1}^{n}|k_j| - 1\right) S\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\wedge e_{-1}\right)\\
&= \left(\sum_{j=1}^{n} q_j - 1\right) S\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\wedge e_{-1}\right)\\
&= SQ\left(e_{k_1}\wedge\ldots\wedge e_{k_n}\wedge e_{-1}\right)
\end{aligned}
$$

since each k_j, $j = 1,\ldots,n$ ($j \neq n+1$) has the same sign as k_j+1, so $|k_j+1|-|k_j| = q_j$, for $j = 1,\ldots,n$ and $q_{n+1} = -1$. Hence the commutator formula is proved on product vectors, so the formula holds on $\mathcal{D}(H) \subset \mathcal{D}$. We have earlier (in section 4.1) shown that $[Q,S] = S$ on $\mathcal{D}$. Then it follows that

$$\left[H - \frac{1}{2}Q(Q-I), S\right] = 0$$

on $\mathcal{D}(H)$, since

$$\left[H - \frac{1}{2}Q(Q-I), S\right] = [H,S] - \frac{1}{2}Q[Q-I,S] - \frac{1}{2}[Q,S](Q-I)$$
$$= SQ - \frac{1}{2}Q(S-0) - \frac{1}{2}S(Q-I)$$
$$= \frac{1}{2}[S,Q] + \frac{1}{2}S = -\frac{1}{2}S + \frac{1}{2}S = 0$$

by use of the above commutation formulas. Moreover, evidently $[H,Q]=0$ on Ω and on product basis vectors $e_{k_1} \wedge \ldots \wedge e_{k_n}$, hence on all of $\mathcal{D}(H)$.

So H have the following decomposition as a direct sum

$$H = \oplus_{q \in \mathbf{Z}} H_q$$

where

$$H_q = H|_{\mathcal{H}_q}$$

Recall from section 1.5 that $\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{q\in\mathbf{Z}}\mathcal{H}_q$. Moreover, we put $\Omega_q = S^q\Omega$, where S^q is well-defined for $q \in \mathbf{Z}$, since S is invertible, $S^{-1} = S^*$, and Ω_q belongs to $\mathcal{H}_q$, since $\Omega_q = S^{q-1}e_0 = S^{q-2}e_1 \wedge e_0 = \ldots = e_{q-1} \wedge e_{q-2} \wedge \ldots \wedge e_2 \wedge e_1 \wedge e_0$, for $q > 0$, and $\Omega_q = (S^*)^{-q-1}e_{-1} = (S^*)^{-q-2}e_{-2} \wedge e_{-1} = \ldots = e_q \wedge e_{q+1} \wedge \ldots \wedge e_{-2} \wedge e_{-1}$, for $q < 0$.

From the commutation relation above, it follows that

$$\left(H - \frac{1}{2}Q(Q-I)\right)\Omega_q = S^q\left(H - \frac{1}{2}Q(Q-I)\right)\Omega = 0$$

since $H\Omega = Q\Omega = 0$. Let now $e_{k_1} \wedge \ldots \wedge e_{k_n}$ be an arbitrary product basis vector in $\mathcal{H}_q$, i.e. $Q(e_{k_1} \wedge \ldots \wedge e_{k_n}) = \sum_{j=1}^n q_j \cdot (e_{k_1} \wedge \ldots \wedge e_{k_n})$, where $\sum_{j=1}^n q_j = \sum_{j=1}^n \mathrm{sign}(k_j) = q$. Then

$$H(e_{k_1} \wedge \ldots \wedge e_{k_n}) = \left(\sum_{j=1}^n |k_j|\right)(e_{k_1} \wedge \ldots \wedge e_{k_n})$$

For $q > 0$ is

$$\sum_{j=1}^n |k_j| \geq \sum_{m=1}^q |k_{j_m}| \geq \sum_{m=1}^{q-1} m$$
$$= \frac{1}{2}(q-1)(q-1+1) = \frac{1}{2}q(q-1)$$

where $k_{j_1}, \ldots, k_{j_m}$ are q arbitrary positive (or zero) indices in increasing order, i.e. $0 \leq k_{j_1} < k_{j_2} < \cdots < k_{j_m}$, it is indeed possible to choose q such indices, since there are exactly q more positive (or zero) than negative indices $k_1, \ldots k_n$ all different from each other (or $e_{k_1} \wedge \ldots \wedge e_{k_n} = 0$).

For $q < 0$ we have

$$\begin{aligned} \sum_{j=1}^{n} |k_j| &\geq \sum_{m=1}^{-q} |k_{j_m}| \geq \sum_{m=1}^{-q} m \\ &= \frac{1}{2}(-q)(-q+1) = \frac{1}{2}q(q-1) \end{aligned}$$

where $k_{j_1} < k_{j_2} < \cdots < k_{j_{-q}} < 0$ are chosen arbitrarily between the at least $-q$ negative indices $k_1, \ldots k_n$, which is possible since there are $-q$ more negative than positive (or zero) indices, all different from each other.

For $q = 0$ is

$$\sum_{j=1}^{n} |k_j| \geq 0 = \frac{1}{2}q(q-1)$$

Hence

$$\begin{aligned} H\left(e_{k_1} \wedge \ldots \wedge e_{k_n}\right) &\geq \frac{1}{2}q(q-1)\left(e_{k_1} \wedge \ldots \wedge e_{k_n}\right) \\ &= \frac{1}{2}Q(Q-I)\left(e_{k_1} \wedge \ldots \wedge e_{k_n}\right) \end{aligned}$$

Thus $H \geq \frac{1}{2}Q(Q-I) = \frac{1}{2}q(q-1)$ on product basis vectors in $\mathcal{H}_q$.

Therefore the representation of the Virasoro algebra, given above, restricted to $\mathcal{H}_q$, is caracterized by the label $\left(\frac{1}{8} + \frac{1}{2}q(q-1), 1\right)$, where $\frac{1}{8} + \frac{1}{2}q(q-1)$ is the minimal energy eigenvalue of the new energy operator H_h on $\mathcal{H}_q$ (corresponding to the sector vacuum Ω_q). The earlier mentioned label then corresponds to the sector $\mathcal{H}_0$. Observe that the representations corresponding to q and $-q+1$ gives rise to the same label (of course on different sectors), they are therefore unitarily equivalent. Since D_k map $\mathcal{H}_q$ into $\mathcal{H}_q$ and $[H, D_k] = kD_k$, for $k \in \mathbf{Z} \setminus \{0\}$, by the earlier derived commutation relations (or the complexification of formula (1.17)), it follows that $HD_k\Omega_q = D_kH\Omega_q + kD_k\Omega_q = (\frac{1}{2}q(q-1) + k)D_k\Omega_q$. Hence the energy of $D_k\Omega_q \in \mathcal{H}_q$ has energy stricly less than the sector vacuum $\Omega_q \in \mathcal{H}_q$ for any k negative, which is a contradiction, or $D_k\Omega_q = 0$. Whence $D_k\Omega_q = 0$, for any negative $k \in \mathbf{Z}$.

We have analysed the diffeomorphism group $Diff^+(S^1)$, on a Lie algebra level, as a unitary group, by use of the spin representation, treated in section 1.5. Thereby we have constructed positive energy representations of the Virasoro algebra in the charge sectors $\mathcal{H}_q$. These representations have all central charge $c = 1$ and minimal energy $h + \frac{1}{2}q(q-1)$, respectively, where $h = \frac{1}{8}$, corresponding to the energy operator $H_h = H + h$.

There are several ways in which $Diff^+(S^1)$ can act on $\mathcal{H} = L_2(S^1)$, we have treated one realization of $Diff^+(S^1)$ above and will treat another in the next section, namely $Diff^+(S^1)$ as a symplectic group.

4.3 The diffeomorphism group $Diff^+(S^1)$ as a symplectic group.

In section 4.2 we considered $Diff^+(S^1)$ as a subgroup of the restricted unitary group acting on $\mathcal{H} = L_2(S^1)$, and used again the spin representation, constructed in section 1.4 and 1.5, to obtain a positive energy projective representation on a Lie algebra level, in the anti-symmetric Fock Hilbert space, which turned out to be a sequence of realizations of the Virasoro algebra with central charge $c = 1$. As mentioned earlier this is not the only action of $Diff^+(S^1)$ on the Hilbert space $\mathcal{H} = L_2(S^1)$, in fact there are several other possible actions (see [Ne, p. 411]).

In this section we will make considerations analogous to those of section 4.2. However, we will consider $Diff^+(S^1)$ as a subgroup of the symplectic group acting on a Hilbert space $\mathcal{H}_0^{1/2}$, and use the metaplectic representation, constructed in section 2.3, to achieve a positive energy projective representation, on a Lie algebra level, in the symmetric Fock Hilbert space. This representation turns out to be a realization of the Virasoro algebra with central charge $c = 1$ and lowest weight $h = 0$.

The following exposition is parallel to what we did in section 4.2 and is based on [Lu 2] and [Ne].

Consider the infinite vector space $\mathcal{H}^{1/2}$ of real functions on the unit circle S^1 such that $\sum_{k\in\mathbf{N}} k \cdot |f_k|^2 < \infty$, where $f_k = \langle e_k, f\rangle_{\mathcal{H}}$ is the k'th Fourier component of f with respect to the inner product in $\mathcal{H} = L_2(S^1)$, given in section 4.2. We introduce the semi-inner product on $\mathcal{H}^{1/2}$, as in the case of the spin representation of the special loop group, treated en section 4.1, given in terms of the Fourier components as $\langle f, g\rangle_{1/2} = \frac{1}{2}\sum_{k\in\mathbf{N}} k \cdot (\overline{f}_k g_k + f_k \overline{g}_k)$. Notice that the Fourier components may be complex even though the original functions are real. The only restriction on the Fourier component f_k is that $\overline{f}_k = \overline{\langle e_k, f\rangle}_{\mathcal{H}} = \langle e_{-k}, f\rangle_{\mathcal{H}} = f_{-k}$, since f is real. Observe that the semi-inner product given above is in fact a complex inner product on ℓ^2 (for details see section 4.1).

The semi-inner product, given above, is not a inner product since $\mathcal{H}^{1/2}$ has a one-dimensional null space with respect to the semi-norm arising from the semi-inner product $\langle\cdot,\cdot\rangle_{1/2}$. This null space consists of the constant functions

$f = f_0$ (where f_0 denote the 0'th Fourier component). Hence the quotient space $\mathcal{H}_0^{1/2} = \mathcal{H}^{1/2}/\{f : f = f_0 \in \mathbf{R}\}$ is a Hilbert space, with inner product $\langle \cdot, \cdot \rangle_{1/2}$.

First we define the complex unit operator J on $\mathcal{H}_0^{1/2}$ by

$$J\left(\sum_{k\in\mathbf{Z}\setminus\{0\}} f_k e_k\right) = \sum_{k\in\mathbf{N}} (if_k)e_k + \sum_{k\in\mathbf{N}} (-if_{-k})e_{-k}$$

i.e. J multiply the positive Fourier component f_k, $k \in \mathbf{N}$, by i and the negative Fourier component f_{-k}, $k \in \mathbf{N}$, by $-i$. It follows that $Jf \in \mathcal{H}_0^{1/2}$, for $f \in \mathcal{H}_0^{1/2}$, since

$$\begin{aligned}
\overline{(Jf)} &= \sum_{k\in\mathbf{Z}\setminus\{0\}} \operatorname{sign}(k)(-i)\overline{f_k}\overline{e_k} = \sum_{k\in\mathbf{Z}\setminus\{0\}} \operatorname{sign}(-k)if_{-k}e_{-k} \\
&= \sum_{-k\in\mathbf{Z}\setminus\{0\}} \operatorname{sign}(k)if_k e_k = Jf
\end{aligned}$$

and

$$\sum_{k\in\mathbf{N}} k\left|(Jf)_k\right|^2 = \sum_{k\in\mathbf{N}} k\left|\operatorname{sign}(k)if_k\right|^2 = \sum_{k\in\mathbf{N}} k\left|f_k\right|^2 < \infty$$

The above calculations are of course equivalent to consider J on $\mathcal{H}^{1/2}$ modulo constant functions.

Obviously $J^2 = -1$, by direct computations, and $J^* = -J$, since

$$\begin{aligned}
\langle g, Jf\rangle_{1/2} &= \frac{1}{2}\sum_{k\in\mathbf{N}} k\left(\overline{g_k}if_k + g_k(-i)\overline{f_k}\right) \\
&= \frac{1}{2}\sum_{k\in\mathbf{N}} k\left(\overline{(-Jg)}_k f_k + (-Jg)_k\overline{f_k}\right) \\
&= \langle -Jg, f\rangle_{1/2}
\end{aligned}$$

Thus J introduces a complex structure on the set $\mathcal{H}_0^{1/2}$. The complexification $\mathcal{H}_J^{1/2}$ of $\mathcal{H}_0^{1/2}$ is a Hilbert space with respect to the complex inner product given by

$$\langle f, g\rangle_J = \langle f, g\rangle_{1/2} + i\,\langle Jf, g\rangle_{1/2}$$

Notice that this complex structure is not the usual one. Hence we may write

$$\begin{aligned}\langle f,g\rangle_J &= \frac{1}{2}\sum_{k\in\mathbf{N}} k\overline{f_k}g_k + \frac{1}{2}\sum_{k\in-\mathbf{N}} |k|\,\overline{f_k}g_k \\ &\quad +\frac{i}{2}\sum_{k\in\mathbf{N}} k\overline{(if_k)}g_k + \frac{i}{2}\sum_{k\in-\mathbf{N}} |k|\,\overline{(-if_k)}g_k \\ &= \sum_{k\in\mathbf{N}} k\overline{f_k}g_k\end{aligned}$$

for $f,g\in\mathcal{H}_J^{1/2}$.

Thus we define the bilinear form $\sigma(\cdot,\cdot)$ on $\mathcal{H}_0^{1/2}$ by

$$\sigma(f,g) = Im\,\langle f,g\rangle_J$$

Then

$$\sigma(f,g) = \frac{1}{2i}\left(\sum_{k\in\mathbf{N}} k\overline{f_k}g_k - \sum_{k\in\mathbf{N}} kf_k\overline{g_k}\right) = \frac{i}{2}\sum_{k\in\mathbf{Z}\setminus\{0\}} kf_k\overline{g_k}\,.$$

Moreover, we may rewrite $\sigma(f,g)$ as

$$\begin{aligned}\sigma(f,g) &= \frac{i}{2}\sum_{k\in\mathbf{Z}\setminus\{0\}} kf_k\overline{g_k} = \frac{1}{2}\left\langle \sum_{n\in\mathbf{Z}} f_ne_n, \sum_{k\in\mathbf{Z}} ikg_ke_k\right\rangle_{\mathcal{H}} \\ &= \frac{1}{2}\langle f(\cdot),g'(\cdot)\rangle_{\mathcal{H}} = \frac{1}{4\pi}\int_0^{2\pi} f(\theta)g'(\theta)d\theta\end{aligned}$$

for smooth functions $f,g\in\mathcal{H}_0^{1/2}$. Evidently $\sigma(\cdot,\cdot)$ is a non-degenerated symplectic form on $\mathcal{H}_0^{1/2}$.

The natural action of $Diff^+(S^1)$ on $\mathcal{H} = L_2(S^1)$ is given by

$$(s_\phi f)(\theta) = f(\phi(\theta))$$

where ϕ is a smooth real function such that $\phi(\theta+2\pi) = \phi(\theta)+2\pi$ and $\phi'(\theta)>0$ defining the diffeomorphism $\psi\in Diff^+(S^1)$ by $\psi(e^{i\theta}) = e^{i\phi(\theta)}$, see section 4.2 for further details. In fact, $\psi\to s_\phi$ defines an anti-representation of $Diff^+(S^1)$, since $\psi_2\circ\psi_1\to s_{\phi_1\circ\phi_2}$ and exactly as in section 4.2 one shows that

$$(s_{\phi_2}(s_{\phi_1}f))(\theta) = (s_{(\phi_1\circ\phi_2)}f)(\theta)$$

Notice that s_ϕ is real linear, with respect to J, and is invertible on $\mathcal{H}_J^{1/2}$, with inverse $s_{\phi^{-1}}$. Moreover, $\sigma(\cdot,\cdot)$ is invariant under s_ϕ,

$$\begin{aligned}\sigma(s_\phi f, s_\phi g) &= \frac{1}{4\pi}\int_0^{2\pi} f(\phi(\theta))\frac{d}{d\theta}(g(\phi(\theta)))d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi} f(\tilde{\theta})g'(\tilde{\theta})d\tilde{\theta} = \sigma(f,g)\end{aligned}$$

for all $f, g \in \mathcal{H}_0^{1/2}$, where $\tilde{\theta} = \phi(\theta)$. Hence $s_\phi \in Sp(\mathcal{H}_J^{1/2})$. In fact s_ϕ is real linear with respect to J, and as usual we split s_ϕ into a complex linear part $(s_\phi)_1 = -\frac{1}{2}J[s_\phi, J]_+$ and a complex anti-linear part $(s_\phi)_2 = \frac{1}{2}J[s_\phi, J]$, which becomes Hilbert-Schmidt. So we only have to prove that $[s_\phi, J]$ is Hilbert-Schmidt, then is also $(s_\phi)_2$. The complex structure J only differs from the previous polarization operator, discussed in the beginning of section 4.2 (and there denoted by a J), by the factor i, i.e. $J = iJ_0$, where J_0 is the polarization operator on $\mathcal{H}_J^{1/2}$, that is $J_0 f_\pm = \pm f_\pm$, for $f_\pm \in (\mathcal{H}_J^{1/2})_\pm$ consisting of all vectors in $\mathcal{H}_J^{1/2}$ such that all the negative (+) respectively positive (−) Fourier components vanish (see also the beginning of section 4.1).

Then J, as J_0, is a singular integral operator with kernel $K(\theta,\varphi) = i - \cot\frac{1}{2}(\theta-\varphi)$ (see the discussion in section 4.2). Since the kernel of s_ϕ is given by $\delta(\phi(\theta)-\varphi)$, then the kernel of the commutator $[s_\phi, J]$ is given by

$$\begin{aligned}&\int_0^{2\pi} d\alpha\,(\delta(\phi(\theta)-\alpha)K(\alpha,\beta) - K(\theta,\alpha)\delta(\phi(\alpha)-\beta)) \\ &= K(\phi(\theta),\beta) - K(\theta,\phi^{-1}(\beta))\phi'(\phi^{-1}(\beta))^{-1} \\ &= K(\phi(\theta),\beta) - K(\theta,\phi^{-1}(\beta))(\phi^{-1})'(\beta))\end{aligned}$$

As in the discussion in section 4.2, this kernel is indeed smooth, except possibly on the diagonal $\beta = \phi(\theta)$, however, it is at least continous at the diagonal. By use of a Taylor expansion, for $|\beta - \phi(\theta)|$ sufficiently small, the prospected singularity in the kernel does not appear,

$$\frac{1}{\phi(\theta)-\beta} - \frac{(\phi^{-1})'(\beta)}{\theta - \phi^{-1}(\beta)} = \frac{(\phi^{-1})''(\phi(\theta))}{(\phi^{-1})'(\phi(\theta))}$$

by calculations analogous to those in section 4.2. So the kernel is indeed continous at the diagonal, since ϕ, and then ϕ^{-1}, is a strictly monotonous diffeomorphism. Hence, the kernel is square integrable on $S^1 \times S^1$, which means that the operator it represents, $[s_\phi, J]$, is Hilbert-Schmidt. Therefore $s_\phi \in \mathcal{S}p_2(\mathcal{H}_J^{1/2})$, and we can construct the metaplectic representation of $\mathit{Diff}^+(S^1)$, considered as a symplectic group.

As a basis for the Lie algebra of real vector fields, acting on the Hilbert space $\mathcal{H}_0^{1/2}$, one usually choses $\cos(k\theta)\frac{d}{d\theta}$ and $\sin(k\theta)\frac{d}{d\theta}$. However, it is more convenient to use the ordinary complex structures and thereby introduce the basis $d_k = e_k \frac{d}{d\theta}$, $k \in \mathbf{Z}$, where $e_k(\theta) = e^{ik\theta}$. Of course these operators d_k act in the ordinary complexification $\mathcal{H}_{\mathbf{C}}^{1/2}$ of $\mathcal{H}_0^{1/2}$. Notice that these operators are unbounded, but they have a common maximal domain $\mathcal{D}_{max}$ given by $f \in \mathcal{H}_{\mathbf{C}}^{1/2}$ such that $\|d_k f\|^2_{\mathcal{H}_{\mathbf{C}}^{1/2}} = \|\sum_{n\in\mathbf{Z}} n f_n e_{n+k}\|^2 = \sum_{n\in\mathbf{Z}} n^2 |f_n|^2$ is finite. One shall beware that we are now operating with two different complex structure. Later on we will only consider the (ordinary) imaginary part. Of course, there will be no trouble if the (ordinary) complex linear operators commutes with J. However, this is not quite the case for the basis elements d_k, $[d_k, J] \neq 0$, but the commutator is Hilbert-Schmidt, and is, in fact, of finite rank. Since $d_k J e_n = -|n|\, e_{n+k}$ and $J d_k e_n = \mathrm{sign}(n) \cdot \mathrm{sign}(n+k) \cdot (-|n| \cdot e_{n+k})$ is $[d_k, J]e_n = |n| \cdot e_{n+k} \cdot (\mathrm{sign}(n) \cdot \mathrm{sign}(n+k) - 1)$ and then

$$
\begin{aligned}
[d_k, J]f &= \sum_{n\in\mathbf{Z}\setminus\{0\}} |n| \cdot f_n \cdot e_{n+k} \cdot (\mathrm{sign}(n) \cdot \mathrm{sign}(n+k) - 1) \\
&= \sum_{n\in\mathbf{Z}\setminus\{0\}} n \cdot f_n \cdot e_{n+k} \cdot (\mathrm{sign}(n+k) - \mathrm{sign}(n)) \\
&= \begin{cases} -2 \cdot \sum_{0<n<-k} n \cdot f_n \cdot e_{n+k} & \text{, for } k \leq 0 \\ 2 \cdot \sum_{-k<n<0} n \cdot f_n \cdot e_{n+k} & \text{, for } k > 0 \end{cases} \\
&= -2 \cdot \sum_{n\in\mathbf{N}_k} |n| \cdot f_n \cdot e_{n+k}
\end{aligned}
$$

where $\mathbf{N}_k = \{1, 2, \ldots, -k-1\}$, for $k < -1$, $\mathbf{N}_k = \{-k+1, -k+2, \ldots, -1\}$, for $k > 1$ and $\mathbf{N}_k = \emptyset$, else.

Observe that $d_k^* = -d_{-k}$ with respect to the ordinary complexification on $\mathcal{D}_{max}$, since on basis vectors e_m and e_n we have

$$\langle e_m, d_k e_n \rangle_{\mathbf{C}} = \langle -d_{-k} e_m, e_n \rangle_{\mathbf{C}}$$

where $\langle f, g + i \cdot h \rangle_{\mathbf{C}} = \langle f, g \rangle_{1/2} + i \cdot \langle f, h \rangle_{1/2}$, for $f, g, h \in \mathcal{H}_0^{1/2}$ and $g + i \cdot h \in \mathcal{H}_{\mathbf{C}}^{1/2}$. Hence $d_k^* = -d_{-k}$. Moreover, on any basis vectors e_k is the commutator $[d_m, d_n]$ well-defined and gives $[d_m, d_n] = -i(m-n)d_{m+n}$ since $[d_m, d_n]e_k = -i \cdot (m-n) \cdot d_{m+n}e_k$. The $d_k, k \in \mathbf{Z}$ do of cause fulfil the Jacobi identity

$$[[d_k, d_m], d_n] = [[d_m, d_n], d_k] + [[d_n, d_k], d_m] = 0$$

So by linearity, $\{d_k, k \in \mathbf{Z}\}$ spans a complex vector space which is a Lie algebra, the ordinary complexification of the Lie algebra corresponding to $Diff^+(S^1)$, considered as a symplectic group.

We now complexify the mapping $A \mapsto dU(A)$ which is defined in section 2.3 for $A \in sp_2(\mathcal{H}_J^{1/2})$ only. First we put

$$d_k^r = \cos(k\theta)\frac{d}{d\theta} = \frac{1}{2}(d_k - d_k^*)$$

and

$$d_k^i = \sin(k\theta)\frac{d}{d\theta} = \frac{1}{2i}(d_k + d_k^*)$$

such that $d_k = d_k^r + i \cdot d_k^i$, where the involution $*$ acts on operators on $\mathcal{H}_{\mathbf{C}}^{1/2}$. Notice that $d_k^i = -d_{-k}^i = -(d_k^i)^*$ and $d_k^r = d_{-k}^r = -(d_k^r)^*$. Of course their domains are the maximal ones and they are all equal to $\mathcal{D}_{max}$ defined earlier as those $f \in \mathcal{H}_{\mathbf{C}}^{1/2}$ such that $\sum_{n\in\mathbf{Z}} n^2 |f_n|^2$ is finite, where f_n denotes the n'th Fourier component of f. Both d_k^r and d_k^i are skew-selfadjoint with respect to σ, since

$$\begin{aligned}
\sigma(d_k^r f, g) &= \frac{1}{4\pi}\int_0^{2\pi} \cos(k\theta)f'(\theta)g'(\theta)d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \cos(k\theta)g'(\theta)f'(\theta)d\theta \\
&= \sigma(d_k^r g, f) = -\sigma(f, d_k^r g)
\end{aligned}$$

and quite similarly one sees that $\sigma(d_k^i f, g) = -\sigma(f, d_k^i g)$, for all $f, g \in \mathcal{H}_0^{1/2}$.

The complexification of the above mapping $A \mapsto dU(A)$ is then given by

$$dU(d_k)_{\mathbf{C}} = dU(d_k^r) + idU(d_k^i)$$

We use the abbreviation $D_k = dU(d_k)_{\mathbf{C}}$. These operators D_k, $k \in \mathbf{Z}$, act in $\mathcal{F}_{\mathsf{V}}(\mathcal{H}_J^{1/2})$. From the general theory, derived in section 2.3, it follows that $dU(d_k^r)$ and $dU(d_k^i)$ is well-defined and essentially skew-selfadjoint on $\mathcal{F}_{\mathsf{V}}(\mathcal{H}_J^{1/2})$. Since $(d_k)_2 = \frac{1}{2}J[d_k, J]$ is a Hilbert-Schmidt operator, so are both

$(d_k^r)_2$ and $(d_k^i)_2$, whereby d_k^r and d_k^i both belong to the extended Lie algebra of $sp_2(\mathcal{H}_J^{1/2})$. Therefore D_k, $k \in \mathbf{Z}$ are well-defined in $\mathcal{F}_\vee(\mathcal{H}_J^{1/2})$ and fulfil

$$D_k^* = -dU(d_k^r) + idU(d_k^i) = -dU(d_{-k}^r) - idU(d_{-k}^i) = -D_{-k}$$

on the domain $\mathcal{D}$. Furthermore, we put $H = -iD_0 = -idU(d_0)$, then $H^* = i(-D_0) = H$ on $\mathcal{D}$. Since $(d_0)_2 e_n = \frac{1}{2}J[d_0, J]e_n = 0$, by direct calculation, for any $n \in \mathbf{Z}$, is $(d_0)_2 = 0$ and $(d_0)_1 = d_0$. Alternatively one could define P as the orthogonal projection $\frac{1}{2}(I + iJ)$ onto $(\mathcal{H}_J^{1/2})_- = \operatorname{span}\{e_n : n < 0\}$, the $-i$ eigenspace corresponding to J. Then $(d_0)_2 = Pd_0(1-P) + (1-P)d_0P = 0$ by a straight forward caculation, and $d_0 = (d_0)_1 = Pd_0P + (1-P)d_0(1-P)$. Then

$$\begin{aligned} H(e_{k_1} \vee \ldots \vee e_{k_n}) &= \sum_{j=1}^n e_{k_1} \vee \ldots \vee e_{k_{j-1}} \vee k_j e_{k_j} \vee e_{k_{j+1}} \vee \ldots \vee e_{k_n} \\ &= \left(\sum_{j=1}^n k_j\right)(e_{k_1} \vee \ldots \vee e_{k_n}) \end{aligned}$$

and by the inner product on $\mathcal{F}_\vee(\mathcal{H}_J^{1/2})$ we get

$$\begin{aligned} \langle e_{k_1} \vee \ldots \vee e_{k_n}, He_{k_1} \vee \ldots \vee e_{k_n} \rangle &= \left(\sum_{j=1}^n k_j\right) \sum_{\sigma \in S_n} \prod_{i=1}^n \left\langle e_{\sigma(k_i)}, e_{k_i} \right\rangle_J \\ &= \left(\sum_{j=1}^n k_j\right) \sum_{\sigma \in S_n} \prod_{i=1}^n \left(\delta_{\sigma(k_i)-k_i} \cdot k_i \cdot \chi(k_i)\right) \\ &= \geq 0 \end{aligned}$$

since $\langle e_m, e_n \rangle_J = m \cdot \delta_{m-m}$, for m positive and $\langle e_m, e_n \rangle_J = 0$ for m negative, where $\chi(\cdot)$ denotes the indicator function for $\mathbf{N}$. Notice that it is strictly larger than, in the formula, if and only if all the k_j, $j = 1, \ldots, n$ are strictly positive. Hence H is positive (meaning non-negative). So the representation $d_k \mapsto D_k$ is of positive energy, where $H = -iD_0$ denotes the energy operator, in $\mathcal{F}_\vee(\mathcal{H}_J^{1/2})$ with the common domain $\mathcal{D}$.

By the complexification, linearity and the above, we get the two-point function

$$
\begin{aligned}
\langle D_m\Omega, D_n\Omega\rangle &= \langle \Omega, -D_{-m}D_n\Omega\rangle \\
&= -\left\langle \Omega, \left(dU(d^r_{-m}) + i\cdot dU(d^i_{-m})\right)\left(dU(d^r_n) + i\cdot dU(d^i_n)\right)\Omega\right\rangle \\
&= \frac{1}{2}\mathrm{Tr}_J\left(\left((d^r_n)_2 + i\cdot (d^i_n)_2\right)\left((d^r_{-m})_2 + i\cdot (d^i_{-m})_2\right)\right) \\
&= \frac{1}{2}\mathrm{Tr}_J\left((d_n)_2(d_{-m})_2\right)
\end{aligned}
$$

where the inner product is that of $\mathcal{F}_V(\mathcal{H}_J^{1/2})$ and formula (2.24) of theorem 17 in section 2.3 have been used. Obeserving that

$$
\begin{aligned}
&(d_n)_2(d_{-m})_2e_k \\
&\quad = k(m-k)\cdot(1-\chi(k))\cdot\chi(k-m)(1-\chi(k-m+n))e_{k+n-m} \\
&\qquad + k(m-k)\cdot\chi(k)\cdot(1-\chi(k-m))\cdot\chi(k-m+n)e_{k+n-m}
\end{aligned}
$$

for k positive and where $\chi(\cdot)$ denotes the indicator function for $\mathbf{N}$. Hence

$$
\begin{aligned}
&\frac{1}{2}\mathrm{Tr}_J\left((d_n)_2(d_{-m})_2\right) \\
&\quad = \frac{1}{2}\sum_{k\in\mathbf{Z}\setminus\{0\}}\left\langle \frac{1}{\sqrt{k}}e_k, (d_n)_2(d_{-m})_2\frac{1}{\sqrt{k}}e_k\right\rangle_J \\
&\quad = \frac{1}{2}\sum_{k\in-\mathbf{N}} k(m-k)\chi(k-m)(1-\chi(k-m+n))\delta_{n-m}\chi(k) \\
&\qquad + \frac{1}{2}\sum_{k\in\mathbf{N}} k(m-k)(1-\chi(k-m))\chi(k-m+n)\delta_{n-m}\chi(k) \\
&\quad = \frac{1}{2}\delta_{n-m}\sum_{k=1}^{m} k(m-k)\chi(m) \\
&\quad = \frac{1}{12}m(m^2-1)\delta_{n-m}\chi(m)
\end{aligned}
$$

where we have explicitely used that $\{\frac{1}{\sqrt{k}}e_k\}_{k\in\mathbf{N}}$ form an orthonormal basis for $\mathcal{H}_J^{1/2}$, since $\langle e_m, e_n\rangle_J = m\cdot\delta_{m-n}\cdot\chi(m)$. Notice that $\|D_m\Omega\| = 0$, for m negative. Therefore the Lie algebra cocycle is

$$
\begin{aligned}
\omega(d_m, d_n)_{\mathbf{C}} &= -\frac{1}{2}\mathrm{Tr}_J([(d_n)_2, (d_m)_2]) \\
&= \frac{1}{12} \cdot m \cdot (m^2 - 1) \cdot \delta_{n+m} \cdot (\chi(m) + \chi(-m)) \\
&= \frac{1}{12} \cdot m \cdot (m^2 - 1) \cdot \delta_{n+m}
\end{aligned}
$$

by complexification of formula (2.28) in section 2.3. The positive energy representation is a socalled level one representation, i.e. $(h, c) = (0, 1)$ where $h = 0$ is the minimal energy and $c = 1$ is the central charge.

We end this section by mentioning that there are other symplectic actions of $\mathit{Diff}^+(S^1)$ than the one considered in this section, for those we refer to [Ne, p. 411].

4.4 The boson-fermion correspondance.

In section 4.1 we showed that the mapping $f \to \phi(f)$, $f \in i \cdot slS^1$, provides us with a representation of the canonical commutation relations in the anti-symmetric Fock Hilbert space (see page 177), or equivalently that $e^{if} \to U(e^{if})$ gave a representation of the special loop group SLS^1, which fulfils the Weyl form of the canonical commutation relations (see page 180). As claimed in section 4.1, the first representation mentioned above is unitarily equivalent to the Fock representation $f \to \pi(f)$ on the symmetric Fock space modelled over $\mathcal{H}^{1/2}$, introduced in section 2.1. This remarkable equivalence is well understood by the boson-fermion correspondance. We will not discuss the boson-fermion correspondance in general, but refer to [Mi, p.193-202] and [K-R, p.53-64], however, we will prove it in this particular case.

Theorem 34 *The "sector energy operator" $H_\wedge = H - \frac{1}{2}Q(Q-I)$ is unitarily equivalent to the boson energy operator in each charge sector.*

Proof. We notice that Uhlenbrock, in [Uh], consider a similar correspodence, however, the arguments conserning the mutiplicities are not immediately intelligible, even that the result is correct. Below we bring an alternative argument. Since $[d_0, f]g = f' \cdot g$ it follows that $[d_0, f] = f'$ and hence

$$[H, \phi(f)] = -i[dU_p(d_0), \phi(f)] = -i\phi([d_0, f]) = -i\phi(f')$$

by direct calculation, using formula (1.17) and the fact that $\omega_P(d_0, if) = 0$. Whence

$$[H_\wedge, \phi(f)] = [H - \frac{1}{2}Q(Q-I), \phi(f)] = -i\phi(f')$$

by $[Q, \phi(f)] = 0$ (see page 177). This is the exspected commutator of $H_\wedge$ with $\phi(f)$.

In section 4.2, page 199 we have already shown that

$$[H_\wedge, S] = [H - \frac{1}{2}Q(Q-I), S] = 0$$

and that the energy operator H may be decomposed as $H = \oplus_{q \in \mathbf{Z}} H_q$, where $H_q = H|_{\mathcal{H}_q}$ and $\mathcal{H}_q$ denote the q'th charge sector defined in section 1.5, and that $H_\wedge = H - \frac{1}{2}Q(Q-I) \geq 0$ on each $\mathcal{H}_q$ (see page 201). Especially is $H_\wedge \Omega_q = H_\wedge S^q \Omega = 0$ (see page 200). We notice that the spectrum of H is $\mathbf{N} \cup \{0\}$ and that the spectrum of Q is $\mathbf{Z}$, whence the spectrum of $H_\wedge = H - \frac{1}{2}Q(Q-I)$ is $\mathbf{N} \cup \{0\}$. Moreover, the boson energy operator in $\mathcal{F}_\vee(\mathcal{H}^{1/2})$, given by the second quantization mapping as $-i \cdot dU(d_0)$, has spectrum $\mathbf{N} \cup \{0\}$.

Below we will show that the multiplicities of $H_\wedge$ and that of the boson energy operator are the same, and that $H_\wedge$ and the boson energy operator then are unitarily equivalent.

Let us first consider the boson case. In this case any basis product vector $e_{k_1} \vee \cdots \vee e_{k_n} \in \mathcal{F}_\vee(\mathcal{H}^{1/2})$ with $k_1 \geq \ldots \geq k_n > 0$ and energy $\sum_{j=1}^n k_j = m \in \mathbf{N}$ correspond uniquely to a particular partition of $m \in \mathbf{N}$ into a sum of positive integers, i.e. a set $\{k_1, \ldots, k_n\}$ where $k_1 + \cdots + k_n = m$ and $k_1 \geq \cdots \geq k_n > 0$. Moreover, different partitions corresponds to to orthogonal vectors. The eigenspace $B_m^\vee$ of the boson energy operator corresponding to energy eigenvalue $m \in \mathbf{N}$ are spanned by $e_{k_1} \vee \cdots \vee e_{k_n} \in \mathcal{F}_\vee(\mathcal{H}^{1/2})$, where $\sum_{j=1}^n k_j = m$. Hence the dimension $\dim(B_m^\vee)$ is exactly the number $p(m)$ of partitions of $m \in \mathbf{N}$ into a sum of positive integers. It can be shown that $p(m) = \frac{d^m}{dx^m}\left(\prod_{n=1}^\infty (1-x^n)^{-1}\right)|_{x=0}$, but we will not need this result here.

Let us next consider the fermion case. In this case each product vector $e_{j_1} \wedge \cdots \wedge e_{j_n} \in \mathcal{H}_q$ with $j_1 > \cdots > j_n$ and sector-energy $m = \sum_{l=1}^n |j_l| - \frac{1}{2}q(q-1)$ is uniquely determined by the ordered index-set $(j_1, \ldots, j_n)$, with $j_1 > \cdots > j_n$ and $j_l \in \mathbf{Z}$, $l = 1, \ldots, n$ such that $\mathrm{card}(J_+) - \mathrm{card}(J_-) = q$ and $\sum_{l=1}^n |j_l| - \frac{1}{2}q(q-1) = m$, where $J_+ = \{j \in \mathbf{N} \cup \{0\} : j \in (j_1, \ldots, j_n)\}$ and $J_- = \{j \in -\mathbf{N} : j \in (j_1, \ldots, j_n)\}$. That is, we have an isomorphism between the set of orthonormal basis vectors in $\mathcal{H}_q$ and the set of ordered integer-tuples such that the difference between the number of non-negative elements and the number of negative elements is q and such that $\sum_{l=1}^n |j_l| = m + \frac{1}{2}q(q-1)$. Notice that different index-tuples are maped into orthonormal basis product vectors.

Define the mapping γ from the set of such index-tuples, defined above, into the set of ordered integer-sequences by $\gamma(j_1, \ldots, j_n) = (i) = (i_{-1}, i_{-2}, \ldots)$, where $i_{-l} = j_l$, if j_l is non-negative and the negative elements $i_{-l} \in (i)$ are the negative integers which do not occur in $(j_1, \ldots, j_n)$. The sequence (i) is ordered in decreasing order $i_{-1} > i_{-2} > \cdots$ and $i_{-l-1} = i_{-l} - 1$ from a certain step $(l > n)$. We will shortly write this as $\gamma : J_+ \to I_+ = J_+$ and $\gamma : J_- \to I_- = (-\mathbf{N}) \setminus J_-$. We emphasize that $q = \mathrm{card}(I_+) - \mathrm{card}(I_-^c)$, where

$I_-^c = (-\mathbf{N})\setminus I_- = J_-$, and that there exist a $s_0 \in \mathbf{N}$ such that $i_{-s} = q-s$, for $s \geq s_0$. Integer-sequences fulfilling these demand will be called semi-infinite integer-sequences (of charge q). We do especially have that

$$\begin{aligned}
\sum_{s=1}^{\infty}(i_{-s}-(q-s)) &= \sum_{s=1}^{s_0} i_{-s} - \sum_{s=1}^{s_0}(q-s) \\
&= \sum_{i\in I_+^{(q-s_0)}} i + \sum_{i\in I_-^{(q-s_0)}} i - \sum_{s=q-1}^{q-s_0} s
\end{aligned}$$

where $I_\pm^{(q-s_0)} = \{i \in I_\pm : i \geq q-s_0\}$. The rewriting

$$\sum_{i\in I_-^{(q-s_0)}} i = \sum_{i=-1}^{q-s_0} i - \sum_{i\in I_-^c} i$$

then gives

$$\begin{aligned}
\sum_{s=1}^{\infty}(i_{-s}-(q-s)) &= \sum_{i\in I_+} i - \sum_{i\in I_-^c} i - \sum_{i=1}^{s_0-q} i + \sum_{s=-(q-1)}^{s_0-q} s \\
&= \sum_{i\in I_+\cup I_-^c} |i| - \sum_{s=1}^{|q-1|} s \\
&= \sum_{j\in J_+\cup J_-} |j| - \frac{1}{2}q(q-1) \\
&= m
\end{aligned}$$

Notice that γ in fact defines an isomorphism between the set of index-tuples with charge q and sector-energy m and the set of semi-infinite integer-sequences (i) such that $i_{-1} > i_{-2} > \cdots$, $\operatorname{card}(I_+) - \operatorname{card}(I_-^c) = q$, $i_{-s} = q-s$ for s larger than some $s_0 \in \mathbf{N}$ and $\sum_{s=1}^{\infty}(i_{-s}-(q-s)) = m$. Hence, the dimension of the eigenspace $B_m^\wedge(q)$ of the sector-hamiltonian $H_\wedge|_{\mathcal{H}_q}$ corresponding to the energy eigenvalue $m \in \mathbf{N}$ is equal to the number of different ways one can choose semi-infinite integer-sequences fulfilling the above demands. However, the number of ways one can choose such different semi-infinite integer-sequences is equal to the number $p(m)$ of partitions of m into a sum of positive integers (in non-decreasing order).

Each semi-infinite integers-sequence, for fix $q \in \mathbf{Z}$, can be uniquely written as $(q-1+k_1, q-2+k_2, \ldots, q-n+k_n, q-n-1, q-n-2, \ldots)$ with $k_1 \geq k_2 \geq \cdots \geq k_n > 0$, since $\sum_{s=1}^{\infty}(i_{-s}-(q-s)) = \sum_{s=1}^{n} k_s$. Moreover, one easily get that the number of ways to choose a vector in $\mathcal{H}_q$ with sector-energy 0 is $p(0) = 1$, namely by the choise $\Omega_q = S^q\Omega$. Thus is

$$\dim(B_m^{\wedge}(q)) = p(m) = \dim(B_m^{\vee})$$

and we may define a mapping $U_{q,m} : B_m^{\vee} \to B_m^{\wedge}(q)$ by first choosing orthonormal basis for $B_m^{\vee}$ and $B_m^{\wedge}(q)$, respectively, and then let $U_{q,m}$ map the j'th basis vector into the j'th basis vector, $j = 1, \ldots, p(m)$. By linearity and continuity we extend $U_{q,m}$ to a unitary operator. If we put

$$U_q = \oplus_{m=0}^{\infty} U_{q,m} : \mathcal{F}_{\vee}(\mathcal{H}^{1/2}) = \oplus_{m=0}^{\infty} B_m^{\vee} \to \mathcal{H}_q = \oplus_{m=0}^{\infty} B_m^{\wedge}(q)$$

then we get a unitary operator mapping $\mathcal{F}_{\vee}(\mathcal{H}^{1/2})$ onto $\mathcal{H}_q$ such that $U_q H_{\vee} = H_{\wedge}|_{\mathcal{H}_q} U_q$. Whence $H_{\vee}$ and $H_{\wedge}$ are unitary equivalent in each charge sector $\mathcal{H}_q$ and it follows once more, as stated in the end of section 4.3, that $H_{\wedge}|_{\mathcal{H}_q}$ and $H_{\wedge}|_{\mathcal{H}_{q'}}$ are unitarily equivalent, since $H_{\wedge}|_{\mathcal{H}_q} = U_q U_{q'}^* H_{\wedge}|_{\mathcal{H}_{q'}} U_{q'} U_q^*$. ∎

Chapter 5

Summary, conclusions and outlook

Summary, conclusions and outlook.

As a more personal point of view, we find it amazing how strong a tool the Fock Hilbert spaces are in the considerations of this paper. Evidently one could have used other representation spaces, for example the infinite wedge space, in most of the considerations. However, the Fock Hilbert spaces seems appropriate for our analytical approach and they are essential for our explicit construction of both the spin representation and the metaplectic representation. We also benefit from these spaces when fixing the cocycles by the chosen demand, that one-parameter subgroups are lifted to one-parameter subgroups near the identity. These cocycle formulas are so simple that they may deserve to be called canonical cocycles. We also find that the formulas for the vacuum functionals are remarkabely simple. Our analysis in chapter 1 and 2 is complete on a Lie algebra level and on one-parameter subgroups.

Moreover, the anti-symmetric Fock Hilbert space contains the representation spaces of our constructed unitary highest weight representations of the Virasoro algebra with central charge $c = \frac{1}{2}$, made in section 3.3, by use of the oscillator construction, and with central charge c_m belonging to a discrete series in $[0, 1[$, made in section 3.4, by use of the GKO construction. We emphasize that for $m = 1$ the GKO construction provides us with a representation with central charge $c = \frac{1}{2}$. However, our analysis of the Virasoro algebra is not exhausted, for example is the famours Kac determinant formula not discused. From this determinant formula one can determinate the possible irreducible unitary highest weight representations, namely those where $(c, h) \in \{(c_m, h_m^{(r,s)}) : r, s \in \mathbf{N} \cup \{0\}, 1 \leq s \leq r \leq m+1, m \in \mathbf{N} \cup \{0\}\}$. In section 3.1 we gave a survey on loop algebras, which turned out to be intrinsically connected with the Virasoro algebra through the Sugawara construction. Furthermore, we find the considerations of chapter 4 quite interesting, as we use the spin representation to construct explicit representations of the particular loop group LS^1. It turns out that this construction provides us with a sequence of representations of the CCR-algebra, in the anti-symmetric Fock Hilbert space, indicating the interessting boson-fermion correspondance. In fact, we end up by discussing this particular boson-fermion correspondance, which is very concrete in the considered case. These considerations are also completed.

We do also use both the spin representation and the metaplectic representation to construct explicit representations of the orientation preserving diffeomorphism group of the unit circle $Diff^{+}(S^1)$, in the Fock Hilbert spaces modelled over the one-particle Hilbert space $\mathcal{H} = \mathbf{L}_2(S^1)$ or subspaces thereof. Since the representation theory of the diffeomorphism group is very rich, our considerations here are less complete. Finally, we point out that it is still an open question whatever it is possible to construct representations of the Virasoro algebra with central charge c less than 1, especially $c = \frac{1}{2}$, by use of either the spin representation or the metaplectic representation, in analogy with the constructions made in section 4.2-4.3.

Chapter 6

Summary in danish.

Resumé på dansk.

I dette kapitel bringes en sammenfattende redegørelse af afhandlingens resultater på dansk.

Afhandlingen består dels af kendt stof suppleret med enkelte nye beviser og dels af nogle nye resultater, der er sammenfattet til et selvstændigt hele. Hovedsaglig i starten af hvert afsnit forekommer der nogle historiske bemærkninger og enkelte steder, specielt i afsnit 3.2, nogle eksempler på anvendelser af teorien i fysikken.

Kapitel 1 er hovedsaglig en sammenfattende omskrivning af kendt stof, dog suppleret med enkelte detaljer og beviser, som det ikke har været muligt at finde andre steder.

I *afsnit 1.1* introduceres Fock Hilbertrummene, modelleret over en-partikel Hilbertrum, samt nogle vigtige underrum deraf.

I *afsnit 1.2* beskrives CAR-algebraen og den ækvivalente Clifford algebra samt deres Fock repræsentationer. Det vises at CAR-algebraen, og dermed også Clifford algebraen, er entydig op til *-isomorfier, hvorfor man kan koncentrere sig om deres Fock repræsentationer. Endelig stilles spørgsmålet: For hvilke ortogonale transformationer T er automorfien $\pi(f) \longrightarrow \pi(T^{-1}f)$ af Clifford algebraen unitær implementerbar i Fock repræsentationen ? Spørgsmålet besvares i afsnit 1.4.

I *afsnit 1.3* benyttes det antisymmetriske Fock Hilbertrum og Fock repræsentationen af CAR-algebraen eksplicit til at konstruere anden kvantisering også kaldet Fock-Cook kvantisering.

I *afsnit 1.4* præsenteres "the restricted orthogonal group" $\mathcal{O}_2(\mathcal{H})$ (der findes ikke nogen god oversættelse af dette navn til dansk) samt et valg af Lie algebra. Endvidere konstrueres spinrepræsentationen af $\mathcal{O}_2(\mathcal{H})$ eksplicit på Lie algebra niveau, ved hjælp af anden kvantisering og en generalisering af denne ide. Spinrepræsentationen er en projektiv repræsentation. Konstruktionen bygger på at enhver reel lineær operator kan splittes op i en sum af en kompleks lineær og en kompleks antilineær operator, samt det faktum at den antilineære del af en operator fra $\mathcal{O}_2(\mathcal{H})$, per definition, er Hilbert-Schmidt og at enhver Hilbert-Schmidt operator er i entydig korrespondance med en vektor i det antisymmetriske to-partikel Hilbertrum. Denne diskussion er nært relateret til det i afsnit 1.2 stillede spørgsmål, da svaret er, at automorfien $\pi(f) \to \pi(T^{-1}f)$ er unitær implementerbar i Fock repræsentationen hvis og kun hvis den ortogonale transformation T tilhører $\mathcal{O}_2(\mathcal{H})$. Dette svar var givet af Shale og Stinespring allerede i 1965, [S-S]. Det følger at spinrepræsentationen netop giver de ønskede unitære implementorer. Vores metode er dels konstruktiv og dels benyttes et andet kriterium til at fastlægge fasen, nemlig at en-parameter undergrupper af Lie algebraen løftes til en-parameter undergrupper i en omegn af identiteten i $\mathcal{O}_2(\mathcal{H})$, ved eksponentialafbildningen. Den resulterende cocykel på Lie algebra niveau (svarende til fasen nævnt ovenfor) er så simpel at den fortjener at blive kaldt kanonisk

$$\omega(A, B) = -\frac{1}{2}\mathrm{Tr}([A_2, B_2])$$

hvor A og B er elementer i Lie algebraen og A_2 henholdsvis B_2 betegner deres respektive antilineære del. Til sidst udledes et simpelt udtryk for vacuum funktionalen

$$c(s) = (\det(V_{-s}))^{1/2}$$

hvor $V_{-s} = e^{sA_1}(e^{-sA})_1$ og index 1 refererer til den korresponderende lineære del. En del af beviserne og argumenterne i dette afsnit har jeg ikke kunnet finde andre steder, dog findes de fleste resultater i [Lu 2].

I *afsnit 1.5* indlejres "the restricted unitary group" $\mathcal{U}_2(\mathcal{H}, P)$ i $\mathcal{O}_2(\mathcal{H})$. Herved kan spinrepræsentationen af $\mathcal{U}_2(\mathcal{H}, P)$ defineres og teorien fra afsnit 1.4 tilpasses dette specialtilfælde. Mange beviser og argumenter i dette afsnit har jeg ikke kunnet finde i litteraturen.

I *afsnit 1.6* redegøres for transformationen mellem den anvendte formalisme og en alternativ formalisme som for eksempel Araki [Ar] benytter. Denne transformation er ikke beskrevet andetsteds.

Kapitel 2 redegør for det symmetriske tilfælde. Det viser sig at metoderne fra kapitel 1, hvor det tilsvarende antisymmetriske tilfælde blev behandlet, kan overføres med mindre justeringer. Det er atter den uddybende og sammenfattende form, der sammen med detaljerne i afsnit 2.3, er mine væsentligeste bidrag.

I *afsnit 2.1* betragtes det analoge objekt til *CAR*-algebraen. På grund af manglende norm-kontinuitet er dette objekt ikke en C^*-algebra, men kun en $*$-algebra. Dog kan *CCR*-algebraen konstrueres ud fra denne $*$-algebra, ved hjælp af Stones sætning. *CCR*-algebraen er en C^*-algebra.

I *afsnit 2.2* behandles anden kvantisering helt analogt til det antisymmetriske tilfælde.

I *afsnit 2.3* præsenteres den uendeligdimensionale symplektiske gruppe samt et valg af Lie algebra. Endvidere gives et konstruktivt bevis for "hvis" delen af Shales sætning fra 1962, [Sh]: Automorfien $\pi(f) \to \pi(S^{-1}f)$, hvor S er en symplektisk transformation, er unitær implementerbar hvis og kun hvis S tilhører "the restricted symplectic group", $\mathcal{S}p_2(\mathcal{H})$. Disse unitære implementorer giver den metaplektiske repræsentation, der er en projektiv repræsentation af $\mathcal{S}p_2(\mathcal{H})$. Som i afsnit 1.4 fås en kanonisk cocykel på Lie algebra niveau

$$\omega(A,B) = \frac{1}{2}\mathrm{Tr}([A_2, B_2])$$

hvor A og B tilhører Lie algebraen og A_2 henholdsvis B_2 betegner de respektive antilineære dele. Vacuum funktionalen bliver

$$c(s) = (\det(V_s))^{-1/2}$$

hvor $V_{-s} = e^{sA_1}(e^{-sA})_1$ og index 1 refererer til den korresponderende lineære del. De modsatte fortegn, i forhold til de tilsvarende udtryk i det antilineære tilfælde, skyldes symmetrien af produktvektorerne i de respektive Hilbertrum. Det skal bemærkes at den givne udledning af og beviserne for de fleste resultater i dette afsnit så vidt jeg ved ikke forekommer andetsteds, selv om resultaterne er kendte.

Kapitel 3 er reserveret til en introduktion af loopalgebraer og Virasoro algebraen samt konstruktioner af diverse repræsentationer af Virasoro algebraen for forskellige værdier af den centrale ladning. Disse repræsentationer er transformationer af kendte rent algebraiske konstruktioner til repræsentationer i det antisymmetriske Fock Hilbertrum, med den yderligere topologiske struktur dette rum besidder.

I *afsnit 3.1* beskrives og diskuteres loopalgebraer og deres centrale udvidelser til affine Kac-Moody algebraer samt en yderligere udvidelse også til affine algebraer. Begreberne Cartan delalgebra og højeste-vægt-repræsentationer diskuteres også. Derudover konstrueres Sugawara konstruktionen, der giver generatorerne for en repræsentation af Virasoro algebraen i termer af kvadratiske led af basis elementer for en vilkårlig repræsentation af en affin Kac-Moody algebra. Endelig redegøres der for at de orienteringsbevarende diffeomorfier af enhedscirklen virker som en gruppe af automorfier af den centrale udvidelse af enhver loopalgebra.

I *afsnit 3.2* introduceres og diskuteres Virasoro algebraen indgående samt nogle af dens anvendelser i fysik. Diskussionen af Sugawara konstruktionen fra foregående afsnit følges op og der tages hul på diskussionen om hvilke værdier af den centrale ladning, der er mulig hvis det kræves at repræsentationerne af Virasoro algebraen skal være irreducible unitære højeste-vægt-repræsentationer. Udtrykkene "højeste-vægt" og "laveste-vægt" bruges skiftevis. Konventionelt benyttes udtrykket "højeste-vægt" oftest (af historiske grunde) også i tilfælde hvor udtrykket "laveste-vægt" er bedre beskrivende.

I *afsnit 3.3* focuseres på repræsentationer af Virasoro algebraen med central ladning $c = \frac{1}{2}$. Der konstrueres eksplicitte repræsentationer i det antisymmetriske Fock Hilbertrum i de to essentielt forskellige tilfælde, kaldet Ramond sektoren, henholdvis Neveu-Schwarz sektoren. Derved opnås irreducible unitære laveste-vægt-repræsentationer med positiv energi. Denne oscillatormetode, som bygger på en oscillatoralgebra, kan også benyttes til at konstruere repræsentationer med central ladning større end 1. Den eksplicitte konstruktion på det antisymmetriske Fock Hilbertrum er mig bekendt ikke tidligere beskrevet.

I *afsnit 3.4* overføres den rent algebraiske GKO konstruktion til repræsentationer af Virasoro algebraen i visse underrum af tensor produkter af ladningssektorer af det antisymmetriske Fock Hilbertrum. Herved konstrueres en serie af irreducible unitære højeste-vægt-repræsentationer parametriseret ved de sammenhørende mulige værdier af den centrale ladning og den højeste vægt, hvor den centrale ladning er givet ved

$$c_m = 1 - \frac{6}{(m+2)(m+3)}, \qquad m \in \mathbf{N} \cup \{0\}$$

og den højeste vægt er givet ved

$$h_m(r,s) = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}, \qquad r,s \in \mathbf{N}, 1 \le s \le r \le m+1$$

Efter hvad jeg ved er konstruktionen beskrevet i dette afsnit ikke publiceret tidligere.

Kapitel 4 omhandler anvendelse af de tidligere kapitler, specielt kapitel 1 og 2, på den orienteringsbevarende diffeomorfigruppe på enhedscirklen og på loopgruppen LS^1 over cirkelgruppen, også kaldet loopcirklen. Nogle af resultaterne forekommer i [Lu 2], men med meget få detaljer. Disse samt nogle nye resultater udledes.

I *afsnit 4.1* dekomponeres loopcirklen i et produkt af den specielle loopgruppe SLS^1 og ladningsgruppen. SLS^1 realiseres som en Abelsk undergruppe af $\mathcal{U}_2(\mathcal{H}, P)$ og spinrepræsentationen benyttes til at give eksplicitte repræsentationer af SLS^1 på et Lie algebra niveau. Derved fremkommer der en serie af unitært ækvivalente repræsentationer af CCR-algebraen i det antisymmetriske Fock Hilbertrum, hvorved boson-fermion korrespondancen indiceres (denne behandles senere). I tilfældet med ladningsgruppen viser det sig at elementerne ikke tilhører Lie algebraen, der er benyttet fundamentalt i konstruktionen af spinrepræsentationen, hvorfor de tidligere udviklede metoder ikke kan benyttes. På trods af dette kan unitære implementorer eksplicit konstrueres i dette specialtilfælde. Derved giver produktet af de respektive unitære implementorer, for henholdsvis SLS^1 og ladningsgruppen, projektive repræsentationer af loopcirklen LS^1.

I *afsnit 4.2* studeres den orienteringsbevarende diffeomorfigruppe af enhedscirklen, realiseret som en undergruppe af $\mathcal{U}_2(\mathcal{H}, P)$, hvilket tillader brugen af spinrepræsentationen. Herved opnås en serie af positiv-energi-repræsentationer af Virasoro algebraen med central ladning $c = 1$, parametriseret ved den laveste vægt $h = \frac{1}{8} + \frac{1}{2}q(q-1)$, $q \in \mathbf{Z}$, udtrykt ved spinrepræsentationen i det antisymmetriske Fock Hilbertrum.

I *afsnit 4.3* foretages betragtninger, analoge til de der blev foretaget i afsnit 4.2, denne gang ved at realisere den orienteringsbevarende diffeomorfigruppe af enhedscirklen som en undergruppe af $\mathcal{S}p_2(\mathcal{H})$. Herved kan den metaplektiske repræsentation benyttes, hvorved der opnås en projektiv positiv-energi-repræsentation af Virasoro algebraen med central ladning $c = 1$ og laveste vægt $h = 0$.

I *afsnit 4.4* redegøres der for boson-fermion korrespondancen, som der blev lagt op til i afsnit 4.1. Det her givne bevis optræder ikke andetsteds og er et alternativ til (en generalisering af) det Uhlenbrock giver i [Uh, p. 69-71].

Kapitel 5 er reserveret en kort opsummering og konklusioner, dette er essentielt inkluderet i nærværende resumé.

Kapitel 6 er forbeholdt ovenstående resumé på dansk.

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61/83 "FYSIKS IDEOLOGISKE FUNKTION, SOM ET EKSEMPEL PÅ EN NATURVIDENSKAB - HISTORISK SET".
Projektrapport af: Annette Post Nielsen.
Vejledere: Jens Høyrup, Jens Højgaard Jensen og Jørgen Vogelius.

62/83 "MATEMATISKE MODELLER" - Litteratur på Roskilde Universitetsbibliotek.
En biografi 2. rev. udgave.
Af: Else Høyrup.

63/83 "GREATING ENERGY FUTURES:A SHORT GUIDE TO ENERGY PLANNING".
ENERGY SERIES No. 8.
Af: David Crossley og Bent Sørensen.

64/83 "VON MATEMATIK UND KRIEG".
Af: Berhelm Booss og Jens Høyrup.

65/83 "ANVENDT MATEMATIK - TEORI ELLER PRAKSIS".
Projektrapport af: Per Hedegård Andersen, Kirsten Habekost, Carsten Holst-Jensen, Annelise von Moos, Else Marie Pedersen og Erling Møller Pedersen.
Vejledere: Bernhelm Booss og Klaus Grünbaum.

66/83 "MATEMATISKE MODELLER FOR PERIODISK SELEKTION I ESCHERICHIA COLI".
Projektrapport af: Hanne Lisbet Andersen, Ole Richard Jensen og Klavs Frisdahl.
Vejledere: Jørgen Larsen og Anders Hede Madsen.

67/83 "ELEPSOIDE METODEN - EN NY METODE TIL LINEÆR PROGRAMMERING?"
Projektrapport af: Lone Biilmann og Lars Boye.
Vejleder: Mogens Brun Heefelt.

68/83 "STOKASTISKE MODELLER I POPULATIONSGENETIK"
- til kritikken af teoriladede modeller.
Projektrapport af: Lise Odgård Gade, Susanne Hansen, Michael Hviid og Frank Mølgård Olsen.
Vejleder: Jørgen Larsen.

69/83 "ELEVFORUDSÆTNINGER I FYSIK"
- en test i 1.g med kommentarer.

Af: Albert C. Paulsen.

70/83 "INDLÆRINGS - OG FORMIDLINGSPROBLEMER I MATEMATIK PÅ VOKSENUNDERVISNINGSNIVEAU".
Projektrapport af: Hanne Lisbet Andersen, Torben J. Andreasen, Svend Åge Houmann, Helle Glerup Jensen, Keld Fl. Nielsen, Lene Vagn Rasmussen.
Vejleder: Klaus Grünbaum og Anders Hede Madsen.

71/83 "PIGER OG FYSIK"
- et problem og en udfordring for skolen?
Af: Karin Beyer, Sussanne Blegaa, Birthe Olsen, Jette Reich og Mette Vedelsby.

72/83 "VERDEN IFVØLGE PEIRCE" - to metafysiske essays, om og af C.S Peirce.
Af: Peder Voetmann Christiansen.

73/83 ""EN ENERGIANALYSE AF LANDBRUG"
- økologisk contra traditionelt.
ENERGY SERIES NO. 9
Specialeopgave i fysik af: Bent Hove Jensen.
Vejleder: Bent Sørensen.

74/84 "MINIATURISERING AF MIKROELEKTRONIK" - om videnskabeliggjort teknologi og nytten af at lære fysik.
Projektrapport af: Bodil Harder og Linda Szkotak Jensen.
Vejledere: Jens Højgaard Jensen og Bent C. Jørgensen.

75/84 "MATEMATIKUNDERVISNINGEN I FREMTIDENS GYMNASIUM"
- Case: Lineær programmering.
Projektrapport af: Morten Blomhøj, Klavs Frisdahl og Frank Mølgaard Olsen.
Vejledere: Mogens Brun Heefelt og Jens Bjørneboe.

76/84 "KERNEKRAFT I DANMARK?" - Et høringssvar indkaldt af miljøministeriet, med kritik af miljøstyrelsens rapporter af 15. marts 1984.
ENERGY SERIES No. 10
Af: Niels Boye Olsen og Bent Sørensen.

77/84 "POLITISKE INDEKS - FUP ELLER FAKTA?"
Opinionsundersøgelser belyst ved statistiske modeller.
Projektrapport af: Svend Åge Houmann, Keld Nielsen og Susanne Stender.
Vejledere: Jørgen Larsen og Jens Bjørneboe.

78/84 "JÆVNSTRØMSLEDNINGSEVNE OG GITTERSTRUKTUT I AMORFT GERMANIUM".
Specialrapport af: Hans Hedal, Frank C. Ludvigsen og Finn C. Physant.
Vejleder: Niels Boye Olsen.

79/84 "MATEMATIK OG ALMENDANNELSE".
Projektrapport af: Henrik Coster, Mikael Wennerberg Johansen, Povl Kattler, Birgitte Lydholm og Morten Overgaard Nielsen.
Vejleder: Bernhelm Booss.

80/84 "KURSUSMATERIALE TIL MATEMATIK B".
Af: Mogens Brun Heefelt.

81/84 "FREKVENSAFHÆNGIG LEDNINGSEVNE I AMORFT GERMANIUM".
Specialerapport af: Jørgen Wind Petersen og Jan Christensen.
Vejleder: Niels Boye Olsen.

82/84 "MATEMATIK - OG FYSIKUNDERVISNINGEN I DET AUTOMATISEREDE SAMFUND".
Rapport fra et seminar afholdt i Hvidovre 25-27 april 1983.
Red.: Jens Højgaard Jensen, Bent C. Jørgensen og Mogens Niss.

83/84 "ON THE QUANTIFICATION OF SECURITY":
PEACE RESEARCH SERIES NO. I
Af: Bent Sørensen
nr. 83 er p.t. udgået

84/84 "NOGLE ARTIKLER OM MATEMATIK, FYSIK OG ALMENDANNELSE".
Af: Jens Højgaard Jensen, Mogens Niss m. fl.

85/84 "CENTRIFUGALREGULATORER OG MATEMATIK".
Specialerapport af: Per Hedegård Andersen, Carsten Holst-Jensen, Else Marie Pedersen og Erling Møller Pedersen.
Vejleder: Stig Andur Pedersen.

86/84 "SECURITY IMPLICATIONS OF ALTERNATIVE DEFENSE OPTIONS FOR WESTERN EUROPE".
PEACE RESEARCH SERIES NO. 2
Af: Bent Sørensen.

87/84 "A SIMPLE MODEL OF AC HOPPING CONDUCTIVITY IN DISORDERED SOLIDS".
Af: Jeppe C. Dyre.

88/84 "RISE, FALL AND RESURRECTION OF INFINITESIMALS".
Af: Detlef Laugwitz.

89/84 "FJERNVARMEOPTIMERING".
Af: Bjarne Lillethorup og Jacob Mørch Pedersen.

90/84 "ENERGI I 1.G - EN TEORI FOR TILRETTELÆGGELSE".
Af: Albert Chr. Paulsen.

91/85 "KVANTETEORI FOR GYMNASIET".
1. Lærervejledning
Projektrapport af: Biger Lundgren, Henning Sten Hansen og John Johansson.
Vejleder: Torsten Meyer.

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2. Materiale
Projektrapport af: Biger Lundgren, Henning Sten Hansen og John Johansson.
Vejleder: Torsten Meyer.

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Af: Peder Voetmann Christiansen.

94/85 "TREENIGHEDEN BOURBAKI - generalen, matematikeren og ånden".
Projektrapport af: Morten Blomhøj, Klavs Frisdahl og Frank M. Olsen.
Vejleder: Mogens Niss.

95/85 "AN ALTERNATIV DEFENSE PLAN FOR WESTERN EUROPE".
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Af: Bent Sørensen

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Af: Bjarne Lilletorup.
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Af: Jeppe C. Dyre.

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Af: Bent Sørensen.

99/85 "Der er langt fra Q til R".
Projektrapport af: Niels Jørgensen og Mikael Klintorp.
Vejleder: Stig Andur Pedersen.

100/85 "TALSYSTEMETS OPBYGNING".
Af: Mogens Niss.

101/85 "EXTENDED MOMENTUM THEORY FOR WINDMILLS IN PERTURBATIVE FORM".
Af: Ganesh Sengupta.

102/85 OPSTILLING OG ANALYSE AF MATEMATISKE MODELLER, BELYST VED MODELLER OVER KØERS FODEROPTACELSE OG - OMSÆTNING".
Projektrapport af: Lis Eilertzen, Kirsten Habekost, Lill Røn og Susanne Stender.
Vejleder: Klaus Grünbaum.

103/85 "ØDSLE KOLDKRIGERE OG VIDENSKABENS LYSE IDEER".
Projektrapport af: Niels Ole Dam og Kurt Jensen.
Vejleder: Bent Sørensen.

104/85 "ANALOGREGNEMASKINEN OG LORENZLIGNINGER".
Af: Jens Jæger.

105/85 "THE FREQUENCY DEPENDENCE OF THE SPECIFIC HEAT AF THE GLASS REANSITION".
Af: Tage Christensen.

"A SIMPLE MODEL AF AC HOPPING CONDUCTIVITY".
Af: Jeppe C. Dyre.
Contributions to the Third International Conference on the Structure of Non - Crystalline Materials held in Grenoble July 1985.

106/85 "QUANTUM THEORY OF EXTENDED PARTICLES".
Af: Bent Sørensen.

107/85 "EN MYG GØR INGEN EPIDEMI".
- flodblindhed som eksempel på matematisk modellering af et epidemiologisk problem.
Projektrapport af: Per Hedegård Andersen, Lars Boye, CarstenHolst Jensen, Else Marie Pedersen og Erling Møller Pedersen.
Vejleder: Jesper Larsen.

108/85 "APPLICATIONS AND MODELLING IN THE MATEMATICS CURRICULUM" - state and trends -
Af: Mogens Niss.

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Projektrapport af: Mikael Wennerberg Johansen, Poul Katler og Torben J. Andreasen.
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Af: Bent Sørensen

111/85 JORDEN RUNDT PÅ FLADE KORT".
Projektrapport af: Birgit Andresen, Beatriz Quinones og Jimmy Staal.
Vejleder: Mogens Niss.

112/85 "VIDENSKABELIGGØRELSE AF DANSK TEKNOLOGISK INNOVATION FREM TIL 1950 - BELYST VED EKSEMPLER".
Projektrapport af: Erik Odgaard Gade, Hans Hedal, Frank C. Ludvigsen, Annette Post Nielsen og Finn Physant.
Vejleder: Claus Bryld og Bent C. Jørgensen.

113/85 "DESUSPENSION OF SPLITTING ELLIPTIC SYMBOLS 11".
Af: Bernhelm Booss og Krzysztof Wojciechowski.

114/85 "ANVENDELSE AF GRAFISKE METODER TIL ANALYSE AF KONTIGENSTABELLER".
Projektrapport af: Lone Biilmann, Ole R. Jensen og Arne-Lise von Moos.
Vejleder: Jørgen Larsen.

115/85 "MATEMATIKKENS UDVIKLING OP TIL RENESSANCEN".
Af: Mogens Niss.

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Af: Jeppe C. Dyre.

117/85 "KRAFT & FJERNVARMEOPTIMERING"
Af: Jacob Mørch Pedersen.
Vejleder: Bent Sørensen

118/85 "TILFÆLDIGHEDEN OG NØDVENDIGHEDEN IFØLGE PEIRCE OG FYSIKKEN".
Af: Peder Voetmann Christiansen

119/86 "DET ER GANSKE VIST - - EUKLIDS FEMTE POSTULAT KUNNE NOK SKABE RØRE I ANDEDAMMEN".
Af: Iben Maj Christiansen
Vejleder: Mogens Niss.

120/86 "ET ANTAL STATISTISKE STANDARDMODELLER".
Af: Jørgen Larsen

121/86 "SIMULATION I KONTINUERT TID".
Af: Peder Voetmann Christiansen.

122/86 "ON THE MECHANISM OF GLASS IONIC CONDUCTIVITY".
Af: Jeppe C. Dyre.

123/86 "GYMNASIEFYSIKKEN OG DEN STORE VERDEN".
Fysiklærerforeningen, IMFUFA, RUC.

124/86 "OPGAVESAMLING I MATEMATIK".
Samtlige opgaver stillet i tiden 1974-jan. 1986.

125/86 "UVBY,β - systemet - en effektiv fotometrisk spektralklassifikation af B-,A- og F-stjerner".
Projektrapport af: Birger Lundgren.

126/86 "OM UDVIKLINGEN AF DEN SPECIELLE RELATIVITETSTEORI".
Projektrapport af: Lise Odgaard & Linda Szkotak Jensen
Vejledere: Karin Beyer & Stig Andur Pedersen.

127/86 "GALOIS' BIDRAG TIL UDVIKLINGEN AF DEN ABSTRAKTE ALGEBRA".
Projektrapport af: Pernille Sand, Heine Larsen & Lars Frandsen.
Vejleder: Mogens Niss.

128/86 "SMÅKRYB" - om ikke-standard analyse.
Projektrapport af: Niels Jørgensen & Mikael Klintorp.
Vejleder: Jeppe Dyre.

129/86 "PHYSICS IN SOCIETY"
Lecture Notes 1983 (1986)
Af: Bent Sørensen

130/86 "Studies in Wind Power"
Af: Bent Sørensen

131/86 "FYSIK OG SAMFUND" - Et integreret fysik/historieprojekt om naturanskuelsens historiske udvikling og dens samfundsmæssige betingethed.
Projektrapport af: Jakob Heckscher, Søren Brønd, Andy Wierød.
Vejledere: Jens Høyrup, Jørgen Vogelius, Jens Højgaard Jensen.

132/86 "FYSIK OG DANNELSE"
Projektrapport af: Søren Brønd, Andy Wierød.
Vejledere: Karin Beyer, Jørgen Vogelius.

133/86 "CHERNOBYL ACCIDENT: ASSESSING THE DATA.
ENERGY SERIES NO. 15.
AF: Bent Sørensen.

134/87 "THE D.C. AND THE A.C. ELECTRICAL TRANSPORT IN AsSeTe SYSTEM
Authors: M.B.El-Den, N.B.Olsen, Ib Høst Pedersen, Petr Visčor

135/87 "INTUITIONISTISK MATEMATIKS METODER OG ERKENDELSESTEORETISKE FORUDSÆTNINGER"
MASTEMATIKSPECIALE: Claus Larsen
Vejledere: Anton Jensen og Stig Andur Pedersen

136/87 "Mystisk og naturlig filosofi: En skitse af kristendommens første og andet møde med græsk filosofi"
Projektrapport af Frank Colding Ludvigsen
Vejledere: Historie: Ib Thiersen
Fysik: Jens Højgaard Jensen

137/87 "HOPMODELLER FOR ELEKTRISK LEDNING I UORDNEDE FASTE STOFFER" - Resume af licentiatafhandling
Af: Jeppe Dyre
Vejledere: Niels Boye Olsen og Peder Voetmann Christiansen.

138/87 "JOSEPHSON EFFECT AND CIRCLE MAP."

Paper presented at The International Workshop on Teaching Nonlinear Phenomena at Universities and Schools, "Chaos in Education". Balaton, Hungary, 26 April-2 May 1987.

By: Peder Voetmann Christiansen

139/87 "Machbarkeit nichtbeherrschbarer Technik durch Fortschritte in der Erkennbarkeit der Natur"

Af: Bernhelm Booss-Bavnbek
Martin Bohle-Carbonell

140/87 "ON THE TOPOLOGY OF SPACES OF HOLOMORPHIC MAPS"

By: Jens Gravesen

141/87 "RADIOMETERS UDVIKLING AF BLODGASAPPARATUR - ET TEKNOLOGIHISTORISK PROJEKT"

Projektrapport af Finn C. Physant

Vejleder: Ib Thiersen

142/87 "The Calderón Projektor for Operators With Splitting Elliptic Symbols"

by: Bernhelm Booss-Bavnbek og
Krzysztof P. Wojciechowski

143/87 "Kursusmateriale til Matematik på NAT-BAS"

af: Mogens Brun Heefelt

144/87 "Context and Non-Locality - A Peircean Approach

Paper presented at the Symposium on the Foundations of Modern Physics The Copenhagen Interpretation 60 Years after the Como Lecture. Joensuu, Finland, 6 - 8 august 1987.

By: Peder Voetmann Christiansen

145/87 "AIMS AND SCOPE OF APPLICATIONS AND MODELLING IN MATHEMATICS CURRICULA"

Manuscript of a plenary lecture delivered at ICMTA 3, Kassel, FRG 8.-11.9.1987

By: Mogens Niss

146/87 "BESTEMMELSE AF BULKRESISTIVITETEN I SILICIUM" - en ny frekvensbaseret målemetode.

Fysikspeciale af Jan Vedde

Vejledere: Niels Boye Olsen & Petr Viščor

147/87 "Rapport om BIS på NAT-BAS"
redigeret af: Mogens Brun Heefelt

148/87 "Naturvidenskabsundervisning med Samfundsperspektiv"

af: Peter Colding-Jørgensen DLH
Albert Chr. Paulsen

149/87 "In-Situ Measurements of the density of amorphous germanium prepared in ultra high vacuum"

by: Petr Viščor

150/87 "Structure and the Existence of the first sharp diffraction peak in amorphous germanium prepared in UHV and measured in-situ"

by: Petr Viščor

151/87 "DYNAMISK PROGRAMMERING"

Matematikprojekt af:
Birgit Andresen, Keld Nielsen og Jimmy Staal

Vejleder: Mogens Niss

152/87 "PSEUDO-DIFFERENTIAL PROJECTIONS AND THE TOPOLOGY OF CERTAIN SPACES OF ELLIPTIC BOUNDARY VALUE PROBLEMS"

by: Bernhelm Booss-Bavnbek
Krzysztof P. Wojciechowski

153/88 "HALVLEDERTEKNOLOGIENS UDVIKLING MELLEM MILITÆRE OG CIVILE KRÆFTER"

Et eksempel på humanistisk teknologihistorie
Historiespeciale

Af: Hans Hedål

Vejleder: Ib Thiersen

154/88 "MASTER EQUATION APPROACH TO VISCOUS LIQUIDS AND THE GLASS TRANSITION"

By: Jeppe Dyre

155/88 "A NOTE ON THE ACTION OF THE POISSON SOLUTION OPERATOR TO THE DIRICHLET PROBLEM FOR A FORMALLY SELFADJOINT DIFFERENTIAL OPERATOR"

by: Michael Pedersen

156/88 "THE RANDOM FREE ENERGY BARRIER MODEL FOR AC CONDUCTION IN DISORDERED SOLIDS"

by: Jeppe C. Dyre

157/88 " STABILIZATION OF PARTIAL DIFFERENTIAL EQUATIONS BY FINITE DIMENSIONAL BOUNDARY FEEDBACK CONTROL: A pseudo-differential approach."

by: Michael Pedersen

158/88 "UNIFIED FORMALISM FOR EXCESS CURRENT NOISE IN RANDOM WALK MODELS"

by: Jeppe Dyre

159/88 "STUDIES IN SOLAR ENERGY"

by: Bent Sørensen

160/88 "LOOP GROUPS AND INSTANTONS IN DIMENSION TWO"

by: Jens Gravesen

161/88 "PSEUDO-DIFFERENTIAL PERTURBATIONS AND STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS:

Dirichlet feedback control problems"

by: Michael Pedersen

162/88 "PIGER & FYSIK - OG MEGET MERE"

AF: Karin Beyer, Sussanne Blegaa, Birthe Olsen,
Jette Reich , Mette Vedelsby

163/88 "EN MATEMATISK MODEL TIL BESTEMMELSE AF PERMEABILITETEN FOR BLOD-NETHINDE-BARRIEREN"

Af: Finn Langberg, Michael Jarden, Lars Frellesen

Vejleder: Jesper Larsen

164/88 "Vurdering af matematisk teknologi
Technology Assessment
Technikfolgenabschätzung"

Af: Bernhelm Booss-Bavnbek, Glen Pate med
Martin Bohle-Carbonell og Jens Højgaard Jensen

165/88 "COMPLEX STRUCTURES IN THE NASH-MOSER CATEGORY"

by: Jens Gravesen

166/88 "Grundbegreber i Sandsynligheds-regningen"

Af: Jørgen Larsen

167a/88 "BASISSTATISTIK 1. Diskrete modeller"

Af: Jørgen Larsen

167b/88 "BASISSTATISTIK 2. Kontinuerte modeller"

Af: Jørgen Larsen

168/88 "OVERFLADEN AF PLANETEN MARS"
Laboratorie-simulering og MARS-analoger undersøgt ved Mössbauerspektroskopi.

Fysikspeciale af:

Birger Lundgren

Vejleder: Jens Martin Knudsen
Fys.Lab./HCØ

169/88 "CHARLES S. PEIRCE: MURSTEN OG MØRTEL TIL EN METAFYSIK."

Fem artikler fra tidsskriftet "The Monist". 1891-93.

Introduktion og oversættelse:

Peder Voetmann Christiansen

170/88 "OPGAVESAMLING I MATEMATIK"

Samtlige opgaver stillet i tiden 1974 - juni 1988

171/88 "The Dirac Equation with Light-Cone Data"

af: Johnny Tom Ottesen

172/88 "FYSIK OG VIRKELIGHED"

Kvantemekanikkens grundlagsproblem i gymnasiet.

Fysikprojekt af:

Erik Lund og Kurt Jensen

Vejledere: Albert Chr. Paulsen og
Peder Voetmann Christiansen

173/89 "NUMERISKE ALGORITMER"

af: Mogens Brun Heefelt

174/89 " GRAFISK FREMSTILLING AF FRAKTALER OG KAOS"

af: Peder Voetmann Christiansen

175/89 " AN ELEMENTARY ANALYSIS OF THE TIME DEPENDENT SPECTRUM OF THE NON-STATONARY SOLUTION TO THE OPERATOR RICCATI EQUATION

af: Michael Pedersen

176/89 " A MAXIUM ENTROPY ANSATZ FOR NONLINEAR RESPONSE THEORY"

af : Jeppe Dyre

177/89 "HVAD SKAL ADAM STÅ MODEL TIL"

af: Morten Andersen, Ulla Engström, Thomas Gravesen, Nanna Lund, Pia Madsen, Dina Rawat, Peter Torstensen

Vejleder: Mogens Brun Heefelt

178/89 "BIOSYNTESEN AF PENICILLIN - en matematisk model"

af: Ulla Eghave Rasmussen, Hans Oxvang Mortensen, Michael Jarden

vejleder i matematik: Jesper Larsen
biologi: Erling Lauridsen

179a/89 "LÆRERVEJLEDNING M.M. til et eksperimentelt forløb om kaos"

af: Andy Wierød, Søren Brønd og Jimmy Staal

Vejledere: Peder Voetmann Christiansen
Karin Beyer

179b/89 "ELEVHÆFTE: Noter til et eksperimentelt kursus om kaos"

af: Andy Wierød, Søren Brønd og Jimmy Staal

Vejledere: Peder Voetmann Christiansen
Karin Beyer

180/89 "KAOS I FYSISKE SYSTEMER eksemplificeret ved torsions- og dobbeltpendul".

af: Andy Wierød, Søren Brønd og Jimmy Staal
Vejleder: Peder Voetmann Christiansen

181/89 "A ZERO-PARAMETER CONSTITUTIVE RELATION FOR PURE SHEAR VISCOELASTICITY"

by: Jeppe Dyre

183/89 "MATEMATICAL PROBLEM SOLVING, MODELLING. APPLICATIONS AND LINKS TO OTHER SUBJECTS - State. trends and issues in mathematics instruction

by: WERNER BLUM, Kassel (FRG) og
MOGENS NISS, Roskilde (Denmark)

184/89 "En metode til bestemmelse af den frekvensafhængige varmefylde af en underafkølet væske ved glasovergange

af: Tage Emil Christensen

185/90 "EN NÆSTEN PERIODISK HISTORIE"

Et matematisk projekt

af: Steen Grode og Thomas Jessen

Vejleder: Jacob Jacobsen

186/90 "RITUAL OG RATIONALITET i videnskabers udvikling"

redigeret af Arne Jakobsen og Stig Andur Pedersen

187/90 "RSA - et kryptografisk system"

af: Annemette Sofie Olufsen, Lars Frellesen og Ole Møller Nielsen

Vejledere: Michael Pedersen og Finn Munk

188/90 "FERMICONDENSATION - AN ALMOST IDEAL GLASS TRANSITION"

by: Jeppe Dyre

189/90 "DATAMATER I MATEMATIKUNDERVISNINGEN PÅ GYMNASIET OG HØJERE LÆREANSTALTER

af: Finn Langberg

214|91 "Halløj i æteren" - om elektromagnetisme. Oplæg til undervisningsmateriale i gymnasiet.

Af: Nils Kruse, Peter Gastrup, Kristian Hoppe, Jeppe Guldager

Vejledere: Petr Viscor, Hans Hedal

215|91 "Physics and Technology of Metal-Insulator-Metal thin film structures used as planar electron emitters

by: A.Delong, M.Drsticka, K.Hladil, V.Kolarik, F.Olsen, P.Pavelka and Petr Viscor.

216|91 "Kvantemekanik på PC'eren"

af: Thomas Jessen

217/92 "Two papers on APPLICATIONS AND MODELLING IN THE MATHEMATICS CURRICULUM"

by: Mogens Niss

218/92 "A Three-Square Theorem"

by: Lars Kadison

219/92 "RUPNOK - stationær strømning i elastiske rør"

af: Anja Boisen, Karen Birkelund, Mette Olufsen

Vejleder: Jesper Larsen

220/92 "Automatisk diagnosticering i digitale kredsløb"

af: Bjørn Christensen, Ole Møller Nielsen

Vejleder: Stig Andur Pedersen

221/92 "A BUNDLE VALUED RADON TRANSFORM, WITH APPLICATIONS TO INVARIANT WAVE EQUATIONS"

by: Thomas P. Branson, Gestur Olafsson and Henrik Schlichtkrull